

Frege's Theorem: An Introduction*

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1 Opening

What is the epistemological status of our knowledge of the truths of arithmetic? Are they analytic, the products of pure reason, as Leibniz held? Or are they high-level empirical truths that we know only *a posteriori*, as some empiricists, particularly Mill, have held? Or was Kant right to say that, although our knowledge that ' $5 + 7 = 12$ ' depends essentially upon intuition, it is nonetheless *a priori*? It was with this problem that Gottlob Frege was chiefly concerned throughout his philosophical career. His goal was to establish a version of Leibniz's view: to show that the truths of arithmetic follow logically from premises which are themselves truths of logic. It is this view that we now call *logicism*.

Frege's approach to this problem had a number of strands, but it is simplest to divide it into two: a negative part and a positive part. The negative part consists of criticisms of the views of Mill and Kant, and others who share them. These criticisms, although they are found in a variety of places, mostly occur in the first three chapters of *Die Grundlagen der Arithmetik* (G1). The positive part consists of an attempt to show (not just that but also) how arithmetical truths can be established by pure reason, by actually giving proofs of them from premises which are (or are supposed to be) truths of pure logic. There is thus a purely mathematical aspect of Frege's project. It is this on which I intend to focus here.

Frege was not the first to attempt to show how arithmetical truths can be proven from more fundamental assumptions. His approach, however, was more rigorous and encompassing, by far, than anything that had come before. Leibniz, for ex-

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ample, had attempted to prove such arithmetical truths as ' $2 + 2 = 4$ '. His proofs, however, like those of Euclid before him, rest upon assumptions that he does not make explicit: for example, Leibniz makes free appeal to the associative law of addition, which says that $(a + b) + c = a + (b + c)$; that is, he allows himself freely to 're-arrange' parentheses. But it is essential to any attempt to determine the epistemological status of the laws of arithmetic that we be able precisely to determine upon what assumptions the proofs depend. That is to say, it is essential that the proofs be presented in such a way that, once the premises on which they are to depend have been stated, no additional assumptions can sneak in unnoticed. Frege's idea was to give the proofs within a 'formal system' of logic, in which the permissible inferential steps are explicitly specified, by purely syntactic criteria, so that it becomes no more complicated to determine what assumptions are employed in the proofs than it is, say, to check calculations by long division or, to use a different analogy, to determine whether a move in a game of chess is legal.

Existing systems of logic, deriving from the work of George Boole (and, ultimately from Aristotle), were inadequate for this task, for two reasons. First, the systems were ill-suited to the presentation of proofs. Worse, they were inadequate even to represent the sentences that were contained in many proofs. In particular, it is impossible, in Boole's system, to express sentences containing multiple expressions of generality, such as 'Every horse's head is an animal's head' or, more to the point, 'Every number has a successor'. In a sense, these sentences can be represented within Boole's logic, but not in such a way that one can see, on the basis of that representation, why the former follows from, but does not imply, 'Every horse is an animal' and so not in such a way that proofs even of such simple facts can be carried out.

It was for this reason that Frege was forced to develop a new system of formal logic, which he first presented in *Begriffsschrift* (Bg). His system is, as he frequently points out, adequate for the representation of sentences like those mentioned above; its rules of inference, though limited in number, allow us to carry out the sorts of proofs that could not be formalized in Boole's system. And once a system adequate to the representation of actual mathematical argumentation has been developed, the question of the epistemological status of arithmetic is transformed. It would be too strong to say that it reduces to the question from what assumptions the laws of arithmetic can in fact be proven—for, once such assumptions have been identified, we will be left with the question what the epistemological status of those assumptions is. But nonetheless, the philosophical question takes on a purely mathematical aspect, and mathematical techniques can be brought to bear on it (PCN, opp. 362–3).

It would, I think, be impossible to over-emphasize the importance of the contribution implicit in Frege's approach here. Frege's idea that, through formal logic,

the resources of mathematics can be brought to bear upon philosophical problems pervades contemporary analytic philosophy, having come down to us through the work of Russell, Wittgenstein, Carnap, and others. But its influence is not limited to philosophy. Mathematical logic, as we now have it, is obsessed with the question what can or can not be proven from particular assumptions, and it is only against the background of Frege's system of formal logic—or, at least, a system of formal logic that meets the conditions his was the first to meet—that this question can even be stated in a way that makes it mathematically tractable.

It is in that sense, then, that Frege's approach was more rigorous than existing ones. There is also a sense in which it was more general. When Leibniz attempts to prove the laws of arithmetic, he focuses on such claims as that $2 + 2 = 4$. Frege, however, is interested in even more fundamental arithmetical truths, and for good reason. If it is to be shown that *all* truths of arithmetic are provable from logical laws, then, since there are infinitely many arithmetical truths, this can not be established by literally proving all of them. Rather, some basic arithmetical truths need to be identified, from which all others plausibly follow, and then these basic truths need to be proven. That is to say, arithmetic needs to be *axiomatized*, just as Euclid had axiomatized geometry, and then proofs need to be given of the axioms. Frege was not the first to publish such axioms: that honor is often accorded to Giuseppe Peano, though the historical record makes it clear that Richard Dedekind was actually the first to formulate what are now known as the 'Peano axioms'.¹ Nonetheless, Frege does state axioms for arithmetic (which are interestingly different from Dedekind's and Peano's) and Frege's proofs are directed, in the first instance, at these. He also proves, as Dedekind does, that the axioms are sufficient to characterize the abstract, mathematical structure of the natural numbers—up to isomorphism, as it is said, this being a standard test of the sufficiency of an axiomatization (Heck, 2011b, 2012).

Of course, one can not, in the strictest sense, prove the axioms of arithmetic within a system of pure logic: none of the expressions in the language in which the system is formulated even purport to refer to numbers, so the axioms of arithmetic can not even be written down in such a system. What are required are definitions of the basic notions of arithmetic in terms of logical notions; proofs of the axioms of arithmetic will then become proofs of their definitional translations into the formal, logical system. The mathematical project is thus, in contemporary terminology, to 'interpret' arithmetic in a system of formal logic. To interpret one theory in another—call these the target theory and the base theory, respectively—is to show

¹ The axioms appear explicitly in Dedekind's *Was sind und was sollen die Zahlen?* which was originally published in 1888 and was then translated into English as "The Nature and Meaning of Numbers" (Dedekind, 1902). It is generally accepted, however, that Dedekind formulated his axioms some years previously.

that definitions can be given of the primitive vocabulary of the target theory (in this case, arithmetic) in terms of the primitives of the base theory (in this case, some formal theory of logic), so that definitional transcriptions of the axioms of the target theory become theorems of the base theory.² At the very least, such a result shows that the target theory is consistent if the base theory is: If there were a proof of a contradiction to be had within the target theory, that proof could be replicated within the base theory, by proving the needed axioms of the target theory within the base theory, and then appending the derivation of a contradiction, in the target theory, to them. Hence, if there is no proof of a contradiction to be had within the base theory, there will be none to be had within the target theory, either.

Such interpretability results were well-known to geometers working in Frege's time, and Frege was a geometer by training. Such techniques were among the most frequently used to establish the consistency of various sorts of non-Euclidean geometries: geometries which reject the parallel postulate. Proofs of the consistency of such geometries usually amounted to proofs that they could be interpreted within other theories whose consistency was not in doubt (e.g., within Euclidean geometry itself).

It is worth emphasizing, however, that interpreting arithmetic within a system of formal logic will not necessarily help us to discover the epistemological status of arithmetic—even if we are agreed that the 'system of formal logic' really is a system of logic, i.e., that all of its theorems are analytic truths. The problem can be illustrated as follows. It was Frege's view that, not just arithmetic, but also analysis (that is, the theory of real numbers), is analytic. If so, given a definition of ordered pairs, the theory of Euclidean geometry can be interpreted in analysis, by means of Cartesian co-ordinates. Does it then follow that, on Frege's view, Euclidean geometry must be analytic? That would be unfortunat, for Frege explicitly agreed with Kant that the laws of Euclidean geometry are synthetic *a priori*. But, in fact, there is no inconsistency in Frege's view here. What the interpretability result establishes is just that what *look like* the axioms of Euclidean geometry can be proven within analysis. The question is whether what *looks like* the parallel postulate really does mean what the parallel postulate means. That is, the question just begs to be asked whether the 'definitions' of fundamental geometrical notions in fact capture the meanings of those notions, as we ordinarily understand them, for example, whether 'a point is an ordered triple of real numbers' is a good definition of the word 'point', as it is used in geometry. If it is not, it has not been shown that the truths of Euclidean geometry can be proven in analysis, not if we identify 'truths of Euclidean geometry' by what they *mean* and not just by their orthographic or syntactic structure.

² Interpretations can take a more complicated form, but we need not consider such matters here.

A corresponding question will arise in connection with Frege's definitions of fundamental arithmetical notions. It will, that is to say, be open to a Kantian to question whether the definitions Frege gives do in fact capture the meanings of arithmetical notions as we ordinarily understand them. If they do not, then Frege will not have shown that the truths of arithmetic can be proven within logic, but only that sentences syntactically indistinguishable from the truths of arithmetic can be so proven. And that is not sufficient. So another large part of Frege's project has to be, and is, to argue that the fundamental notions of arithmetic are *logical notions*, that his definitions of them in logical terms are good definitions, in this sense.

Let me summarize the discussion to this point. Frege's philosophical project, to show that the laws of arithmetic are analytic, that they can be known on the basis of reason alone, has a mathematical aspect. His primary goal is to define the fundamental arithmetical notions in terms of notions of pure logic, and then, within a formal system of logic, to prove axioms for arithmetic. That is how Frege will identify 'the basic laws of arithmetic', i.e., the fundamental assumptions on which arithmetical reasoning is founded. As important as this part of the project is, however, it does not answer the epistemological question on its own, for two sorts of questions remain open. First, there is space to question whether Frege's definitions of the basic arithmetical notions really do capture their meaning: whether we have really succeeded in proving the axioms of arithmetic and so in identifying its basic laws. And second, even if that question is answered affirmatively, we will have to ask what the epistemological status of the basic laws is: only if they are themselves truths of logic, analytic truths, will the proof show that the axioms of arithmetic are analytic truths. As we shall see, both questions have been prominent in the literature generated by Frege's project.

2 Frege's System of Formal Logic

The formal logical system that Frege develops in *Begriffsschrift* is, in essence, what we now know as full, impredicative second-order logic. It allows for quantification over 'concepts' and relations—the extra-linguistic references of predicates of various sorts—as well as over objects. The system as it is presented in *Begriffsschrift* does not, however, meet the demands of rigor Frege imposes. In particular, one of its most important rules of inference is never explicitly stated, the rule in question being a rule of substitution. One might think that Frege's omission here is inconsequential: Isn't it just obvious that, if one has a proof of some formula, then the result of substituting various other expressions for the variables which occur in that theorem should also be a theorem? Maybe so, but the claim that substitution

is a valid form of inference is, in the context of second-order logic, an extremely powerful one. In Frege's system, the rule of substitution plays the role played in more modern formulations by the so-called axioms of comprehension: the comprehension axioms characterize the concepts and relations over which the variables of the theory are supposed to range; that is, each of them asserts that a particular concept or relation *exists*. In full second-order logic, one has such a comprehension axiom for every formula of the theory, and the comprehension axioms then jointly assert that every formula $A(x)$ defines a concept, namely, that true of the objects of which the formula is true; that every formula $B(x,y)$ defines a two-place relation; and so forth. And with no comprehension axioms, second-order logic is very weak. Indeed, even if we allow ourselves the so-called predicative comprehension axioms—that is, even if we assume that formulae that do not themselves contain second-order quantifiers define concepts and relations—the resulting logic is still weak, in a well-defined technical sense.

Many philosophers have worried that appeal to impredicative axioms of comprehension introduces a kind of circularity into the characterization of the concepts and relations over which the variables of the theory range. The worry, as it appears in Russell, for example, is that it must be circular to characterize the concepts the theory talks about by quantifying over the concepts the theory talks about. Although Frege never discussed this problem, his response, I think, would have been to say that he does not propose to say what concepts lie within the domain of the theory by means of the comprehension axioms: the domain is to contain *all concepts*, and the question how any one of them might be defined should not be allowed to determine whether it exists; the comprehension axioms simply assert that every formula defines a concept.³ But let me not pursue the matter further here.⁴

Frege's failure to state a rule of substitution in *Begriffsschrift* is thus an important omission. But it is one he remedies in his later presentation of his formal theory in *Grundgesetze der Arithmetik*.⁵ In fact, his presentation of the system there is as rigorous as any before Hilbert and Ackermann's *Grundzüge der theoretischen Logik*, which was published in 1928.

The axioms of arithmetic cannot, however, be proven within full second-order logic alone. It can easily be shown that even the claim that there are two distinct objects is not a theorem of second-order logic; but arithmetic posits the existence, not just of two, but of infinitely many natural numbers. It is this claim that is the central obstacle to any logicist development of arithmetic. Indeed, it is hard

³ The closest Frege comes to discussing this issue is in section 66 of *Grundgesetze* (Gg).

⁴ There is further discussion of this issue in my paper "A Logic for Frege's Theorem" (Heck, 2011g).

⁵ This is all connected with the development of Frege's notion of unsaturatedness (Heck and May, 2013).

to see that there is any more difficult problem facing one who would answer the epistemological question with which we are concerned than to explain the genesis of our knowledge that there are infinitely many numbers, whatever sort of answer she might ultimately want to give.

3 Three Lessons and a Problem

As said above, Frege argues, in the first three chapters of *Die Grundlagen*, that Kantian and empiricist philosophies of arithmetic will not do. He comes away with three lessons on which he proposes to base his own view, and with one very large problem.

The first lesson is that the natural numbers are to be characterized, as Leibniz suggested, by defining ‘zero’ and ‘increase by one’. This is clear enough, and we shall see below how Frege intends to define these notions.

The second lesson is harder to understand: Frege puts the point by saying that “the content of a statement of number is an assertion about a concept” (G1, §55). What he means is that number is not ascribed, strictly speaking, to *objects*. Suppose I were to ask, pointing to a pile of playing cards, “How many?” You might answer by counting the cards and telling me that there are 104. But then again, you might answer 2. It all depends upon how you take my question, which contains an ambiguity that can be resolved by asking: How many what? How many cards? Or how many complete packs? But then, it would seem, the number 104 is ascribed not to the pile of cards, to their aggregate (or their ‘fusion’), that somewhat scattered object whose parts are all and only the parts of the cards. The number 2 can as justly be ascribed to that aggregate. (The packs have the same parts the cards do). One might conclude from this that ascriptions of number are subjective, that they essentially depend upon our way of regarding the aggregate to which number is assigned. But there is an alternative: to say, with Frege, that number is ascribed to a *concept*, in this case, either to the concept *card on the table* or to the concept *complete pack of cards on the table*.

That having been said, it is overwhelmingly natural to suppose that numbers are a kind of higher-order property, that they are ‘properties of concepts’. For example, 0 would be the property a concept has if nothing falls under it. (So the concept *disco album in my collection* has the property zero, since I do not own any disco albums.)⁶ A concept will have the property 1 if there is an object which falls under it, and every object which does fall under it is identical with that one

⁶ This was true when I originally wrote this paper, but a student who read it helpfully sent me four disco albums: one on vinyl; one a cassette; one on 8-track; and one on CD. Just in case it wasn’t clear what I meant by ‘album’.

(so the concept *object identical with George Clinton* would have the property 1, since George Clinton falls under it, and every object that does is identical with him). And a concept will have the property $n + 1$ if there is an object, x , that falls under it and if the concept *object that falls under the original concept, other than x* , has the property n . Thus, the concept *object that is identical with either George Clinton or James Brown* will have the property $1 + 1$ (i.e., 2), for there is an object falling under it, namely, the Godfather of Soul, such that the concept *object that is identical with either George Clinton or James Brown, other than the Godfather of Soul* (that is, James Brown), has the property 1, for this is just the same concept as *object identical with George Clinton*, described in other words.

Frege discusses this sort of proposal in §§55–61 of *Die Grundlagen*. It is not entirely clear why he rejects it, but it seems plausible that his reason, ultimately, is that such an account will not enable us to prove the axioms of arithmetic—not, that is, without some further assumptions (Heck, 2011f, §6.2). Suppose that there were exactly two objects in the world, call them George and James. Then there will be a concept that has the property 0; there will be others that have the property 1; and there will be one that has the property $1 + 1$. But there will be no concept that has the property $1 + 1 + 1$: for there to be such a concept, there would have to be one under which three objects fell and, by hypothesis, there are only two objects in existence. Nor would any concept have the property $1 + 1 + 1 + 1$, for the same reason. Just how one wants to describe the situation here is a delicate question: one can either say that there is no number $1 + 1 + 1$ or one can say that $1 + 1 + 1$ and $1 + 1 + 1 + 1$ turn out, in this case, to be the same. Either way, though, there will be only finitely many numbers, and the laws of arithmetic cannot be proven.

Of course, there are not just two objects in existence: but a similar problem will arise if there are only finitely many. The situation can be remedied if we simply assume, as an axiom, that there are infinitely many objects: this is the course Russell and Whitehead take in *Principia Mathematica* (Whitehead and Russell, 1925). Frege would not have had any interest in this sort of ‘solution’, though. As said, the really hard problem facing the epistemologist of arithmetic is to account for our knowledge that there are infinitely many numbers. It is hard to see how assuming that we know that there are infinitely many things of some *other* sort is supposed to help. Proving the laws of arithmetic from such an assumption simply leaves us with the question of its epistemological status, and that is no advance, since the epistemological status of just this assumption was the hard problem with which we started. Moreover, if the objects we assume to exist are supposed to be physical objects, it seems unlikely that the claim that there are infinitely many of these is one logic could establish: indeed, one might well wonder whether it is even true.

The third lesson, then, is supposed to be that, despite the fact that an ascrip-

tion of number makes an assertion about a concept, numbers themselves are *not* properties of concepts: they are *objects*. To put the point grammatically, number-words are not predicates of predicates, but proper names. (It should, all along, have seemed strange to speak of zero as being a property of a concept, to say such things as that the concept *disco album in my collection* has the *property* zero!)

One might wonder how these two doctrines—that numbers are ascribed to concepts and that numbers are objects—can be jointly held. The answer is that we need only insist that the most fundamental sort of expression that names a number is one of the form ‘the number belonging to the concept *F*’, and that this is a proper name. Ascriptions of number, such as ‘There are 102 cards on the table’, then get re-cast as identity-statements, e.g.: the number belonging to the concept *card on the table* is identical with 102. This does, in an obvious sense, contain an assertion about a concept, but the number 102 does not appear as a *property* of a concept.

This, however, leaves Frege with a problem. He has denied that arithmetic is either synthetic *a priori* or *a posteriori*, partly on the ground that numbers are not given to us either in perception or in intuition. How then *are* they given to us? What, so to speak, is the mode of our cognitive access to the objects of arithmetic? Frege’s way of answering this question is subtly to change it, writing:

Only in the context of a proposition does a word mean something. It will therefore suffice to explain the sense of a proposition in which a number-word occurs. . . . In our present case, we have to define the sense of the proposition ‘the number which belongs to the concept *F* is the same as that which belongs to the concept *G*’; that is to say, we must reproduce the content of this proposition in other terms, avoiding the use of the expression ‘the number which belongs to the concept *F*’.
(Gl, §62)

It is here that Frege makes ‘the linguistic turn’. It is not, of course, that no philosopher before him had ever been concerned with language; Locke, for example, was obsessed with language and argued, repeatedly, that various sorts of philosophical problems were the results of illusions fostered by misunderstandings of language. (Perhaps Locke was the first logical positivist.) What is original about Frege’s approach is that the epistemological problem with which he begins is transformed into a problem *in the philosophy of language*, not so that it can be discarded, but so that it can be solved.

The answer to the epistemological question that is implicit in Frege’s treatment of it is that our cognitive access to numbers may be explained in terms of our capacity to refer to them, in terms of a capacity to denote them by means of expressions we understand. Of course, if the capacity to refer to an object by means of a proper name itself depended upon our having, or at least being able to have, perceptions

or intuitions of it (or of other objects of its kind), no actual benefit would accrue from reconceiving the problem in this way. But it is precisely to deny that the explanation must proceed in such terms that Frege invokes the ‘context principle’, the claim that the meaning of an expression can be explained by explaining the meanings of complete propositions in which it occurs. The goal, in this case, will be to give the definition of ‘the number belonging to the concept F is the same as that belonging to the concept G ’ in terms of concepts themselves belonging to pure logic. If that should be possible, it will follow that there is, so to speak, a purely logical route to an understanding of expressions that refer to numbers, that is, to a capacity to refer to them and so to a capacity for cognitive access to them.

Frege notes, quoting from Hume, that a criterion for sameness of number is ready to hand.⁷ Suppose, for example, that I want to establish that the number of plates is the same as the number of glasses. One way, of course, would be to count them, to assign numbers to the concepts *plate on the table* and *glass on the table* and then to see whether these numbers are the same. But there is another way: I can pair off the plates and the glasses, say, by putting one and only one glass on each plate, and then see whether each plate ends up with a glass on it and whether there are any glasses left over. That is to say, I can attempt to establish a ‘one-one correlation’ between the plates and the glasses: If there is a one-one correlation to be had, the number of plates is the same as the number of glasses; otherwise not.

Indeed, as Frege is fond of pointing out, the process of counting itself relies upon the establishment of one-one correlations (Gg, v. 1, §108).⁸ To count the plates is to establish a one-one correlation between the plates and an initial segment of the natural numbers, beginning with 1; the last number used is then assigned to the concept *plate on the table* as its number. Why, indeed, does the fact that the same number is assigned by this process to the concepts *plate on the table* and *glass on the table* show that the number of plates is the same as the number of glasses? Because, says Frege, if there is a one-one correlation between the plates and the numbers from 1 to n , and another between the glasses and the numbers from 1 to m , then there will be a one-one correlation between the plates and the glasses if, and only if, n is m .

The notion of one-one correlation can itself be explained in logical terms (assuming, that is, that we accept Frege’s claim that the general theory of concepts and relations, as developed in second-order logic, does indeed count as logic). For a relation R to be a one-one correlation between the F s and the G s is for the following two conditions to hold:

⁷ It is by now widely appreciated that Hume did not have the criterion in mind that Frege attributes to him.

⁸ For further discussion of this matter, see my paper “Cardinality, Counting, and Equinumerosity” (Heck, 2011a).

1. The relation is one-one, that is, no object bears R to more than one object, and no object is borne R by more than one object.
2. Every F bears R to some G and every G is borne R by some F .

In contemporary symbolism:

$$\forall x \forall y \forall z \forall w [Rxy \wedge Rzw \rightarrow x = z \equiv y = w] \wedge \\ \forall x [Fx \rightarrow \exists y (Rxy \wedge Gy)] \wedge \forall y [Gy \rightarrow \exists x (Rxy \wedge Fx)]$$

And now we can say that *there is* a one-one correspondence between the F s and the G s if *there is* a relation R that satisfies these conditions. Say that the F s and G s are 'equinumerous' if so. The definition of numerical identity on which Frege settles is then:

The number of F s is the same as the number of G s if, and only if, the F s and the G s are equinumerous.

This principle we shall call *HP*, for "Hume's Principle", since Frege cites Hume when he introduces it (not because anyone thinks Hume really had this in mind).

4 The Caesar Problem and Frege's 'Solution'

The status of this explanation of 'the number of F s' is one of the major open problems with which Frege's philosophy of arithmetic leaves us. It is worth emphasizing, however, that the problem is more general than whether it provides us with a purely logical route to an apprehension of numbers. Frege's idea is that a capacity for reference to abstract objects can, in general, be explained in these sorts of terms. Thus, he thinks, our capacity to refer to directions can be explained in terms of our understanding of names of directions, these in turn explained by means of a principle analogous to HP, namely:

The direction of the line a is the same as the direction of the line b if, and only if, a is parallel to b .

This does not promise a purely logical route to an apprehension of directions, on Frege's view, since lines are given to us only in intuition. But that does not matter: the directions of lines are *not* given in intuition (or so he claims), so there is a corresponding problem about how they are given to us, a problem to which he offers a corresponding solution (Gl, §§64–5). This sort of answer to the question how we are to explain our capacity to refer to abstract objects thus generalizes in a natural way, and the exploration of its strengths and weaknesses has occupied a number of

philosophers in recent years.⁹ Oddly enough, however, Frege ultimately rejects the claim that HP actually does suffice to explain names of numbers. The stated reason is that the definition “will not decide for us whether [Julius Caesar] is the same as the [number of Roman emperors]—if I may be forgiven an example which looks nonsensical”.¹⁰ HP tells us what statements of the form ‘the number of *F*s is the same as the number of *G*s’ mean; but it utterly fails to tell us what statements of the form ‘*q* is the number of *G*s’ mean, except when ‘*q*’ is an expression of the form ‘the number of *F*s’ (Gl, §66). It is far from obvious, though, why this is supposed to be a problem. I myself have come to the conclusion that it is not one problem, but many (Heck, 2011e,f).

First of all, as Frege notes, we do not seem to be in any doubt about Caesar: whatever numbers might be, he’s not one of them; *a fortiori*, he’s not the number of Roman emperors. And it’s hard to see what the basis of this knowledge could be, if it was not somehow contained in our understanding of names of numbers. (Surely it isn’t an *empirical* fact that Caesar isn’t a number!) But if it is, then the offered explanation of names of numbers must, at least, be incomplete, since it does not capture this element of our understanding of them.

The second worry can be seen as arising out of this one. When I said above that Caesar is not a number, I used the concept of number: And if we understood the concept of number, it would be easy to answer the question whether some object, call it *a*, was the number of *F*s. For either *a* is a number or it is not. If it is not, it certainly isn’t the number of *F*s; and, if it is, then it’s the number of *G*s, for some *G*, so HP will tell us whether it is the number of *F*s. Moreover, it seems clear that what Frege needs to explain is not just our capacity to refer to individual numbers, but our understanding of the concept of number generally. Given HP, the natural way to try to do so is to say that something is a number if there is a concept whose number it is. But if we spell that out, we find that what we have said is that *a* is a number if, and only if, there is a concept *F* such that *a* is the number of *F*s. And as said, HP simply fails to explain what ‘*a* is the number of *F*s’ is supposed to mean, except when ‘*a*’ is of the form ‘the number of *G*s’, in which case it is obvious that *a* is a number! Unless there is some other way to define the concept of number, then, we are without any understanding even of *what it means* to say that Caesar is not a number.

⁹ See, for example, Crispin Wright’s *Frege’s Conception of Numbers as Objects* (Wright, 1983), Bob Hale’s *Abstract Objects* (Hale, 1988), and the papers collected in *The Reason’s Proper Study* (Hale and Wright, 2001) and the references therein. I’ve been concerned with the issue myself (Heck, 2011i,c), as well.

¹⁰ Frege is actually discussing the definition of directions here, but it is clear that it is meant to apply, *mutatis mutandis*, to the case of numbers and HP.

All of this having been said, I think we can now begin to see what the problem with HP—considered as a definition or explanation of names of numbers—is supposed to be. The goal, recall, was to explain our cognitive access to numbers by explaining our capacity to refer to them and to do that by defining, or explaining, names of numbers in purely logical terms. But now consider HP again:

The number of *F*s is the same as the number of *G*s if, and only if, the *F*s and the *G*s are equinumerous.

Focus especially on its left-hand side. This looks like an identity-statement. But what reason do we have to think it really is one? Why think that 'the number of *F*s', as it occurs there, is a proper name at all? Why suppose that the sentence has the *semantic* (as opposed simply to orthographic) structure of an identity-statement? Why not say instead that 'the number of *F*s is the same as the number of *G*s' is just an incredibly misleading way of writing 'the *F*s are equinumerous with the *G*s'? If, in fact, the sentence did have the semantic structure of an identity-statement, it would have to be legitimate to replace 'the number of *F*s' with names of other kinds, for example, 'Caesar', and so to consider such sentences as 'Caesar is the number of *G*s'. Or again, it must be permissible to replace 'the number of *F*s' with a variable and so to consider such 'open' sentences as '*x* is the number of *G*s' and to ask whether they are true or false when the variable takes various objects, such as Caesar, as its value. But what the above observations show is that HP, on its own, just doesn't explain what sentences like this are supposed to mean. And that calls into doubt whether it really does explain *identity-statements* containing *names of numbers*.

This problem is of particular importance within Frege's philosophy. I mentioned earlier that one of his central goals is to explain the genesis of our knowledge that there are infinitely many numbers: One of his central reasons for insisting that numbers are objects is that only then can this claim be proven. Frege's strategy for proving it is to argue thus: let 0 be the number belonging to the concept *object that is not the same as itself*; it is clear enough that 0 is indeed that number. Let 1 be the number belonging to the concept *object identical with 0*; 2, the number belonging to the concept *object identical with either 0 or 1*; and so forth. Again, it is clear enough that 1 and 2 are indeed these numbers; and only if we treat numbers as objects can we speak of such concepts as these. But, when we formalize Frege's argument within second-order logic, the expression 'the number belonging to the concept *F*' will be cashed as: the number of *x* such that *Fx*. And thus, the definitions of 0, 1, and 2 become:

0 is the number of *x* such that $x \neq x$

1 is the number of *x* such that $x = 0$

2 is the number of x such that $x = 0$ or $x = 1$

And so forth. And when we replace '0' with its definition in the second line, we get:

1 is the number of x such that $x =$ the number of y such that $y \neq y$

So Frege's definition of 1 contains an open sentence of the form ' x is the number of F s'—precisely the sort of sentence HP is impotent to explain.

It is a nice question whether there is a way around this problem. For present purposes, however, let me just record it. Whatever its ultimate resolution, Frege himself declared it insoluble and so abandoned the attempt to use HP as an explanation of names of numbers. In its place, he installs an *explicit* definition of names of numbers in terms of extensions of concepts:

The number belonging to the concept F is: the extension of the concept *concept equinumerous with the concept F* .

Extensions here can be thought of as sets: the number of F s is thus the set of all concepts that are equinumerous with the concept F . So 0 will turn out to be the set of all concepts under which nothing falls; 1, the set of all 'singly instantiated' concepts; 2, the set of all 'doubly instantiated' concepts; and so on and so forth.

This move, however, turned out to be disastrous. In order to give this definition within his formal system of logic and prove the laws of number from it, Frege requires some axioms that tell us what we need to know about the extensions of concepts. And, if logicism is to be established, these axioms need, at least plausibly, to be logical truths. There is an eminently natural axiom to hand. Talk of the 'extensions' of concepts is governed by a principle of extensionality, that the concepts F and G have the same extension only if every F is a G and every G is an F —only if, that is to say, the F s just are the G s. And surely, one might say, every concept must have an extension. So the necessary axiom can be taken to be the following:

The extension of the concept F is the same as the extension of the concept G iff every F is a G and every G is an F .

Taken as a definition or explanation of names of extensions, this principle would suffer from problems similar to those afflicting HP. Frege therefore does not so take it, but regards it as an axiom, saying in *Die Grundlagen*, simply that he "assume[s] it known what the extension of a concept is" (Gl, §68, note).

Unfortunately, however, this axiom, which in *Grundgesetze* becomes Frege's Basic Law V, suffers from a fatal problem of its own: in the context of full second-order logic, it is inconsistent since, as Russell showed, paradox arises when we

consider the concept *object that does not fall under any concept whose extension it is* and ask whether *its* extension falls under it. If it does, it doesn't; if it doesn't, it does. Whoops.

5 Frege's Theorem

Until the mid-1980s, the story would have ended here. Frege does show that, given Basic Law V and the definition of numbers as extensions of certain concepts, the axioms of arithmetic can all be proven. But this fact is uninteresting, since anything can be proven in an inconsistent theory. The axioms of arithmetic can be proven, but then so can their negations.

A closer look at the structure of Frege's proofs reveals something interesting, however. Although Frege abandons HP as a definition of names of numbers, he does not abandon it entirely. He continues to assign it a central role within his philosophy. Frege is not entirely explicit about why, but the reason seems, to me anyway, to be that he did not regard HP as wrong but rather as incomplete. The concept of number really is, according to Frege, intimately bound up with the notion of one-one correlation; but one cannot use this observation to define names of numbers via HP. Still, any acceptable definition must be compatible with HP, must yield it as a relatively immediate consequence. Completing the definition then takes the form of providing an explicit definition from which HP can be recovered. And, indeed, the very first thing Frege proves, once his explicit definition has been given, is that HP follows from it GI, §73.

In his paper "Frege's Theory of Number", Charles Parsons (1995) observes that, once Frege has proven HP, the explicit definition quietly drops out of sight—and, with it, all further references to extensions in Frege's proof. That is to say, the proof proceeds in two quite separate stages: First, there is a proof of HP from Basic Law V and the definition of names of numbers in terms of extensions; Second, there is a proof of the axioms of arithmetic from HP within pure second-order logic. Parsons's observation did not cause much of a stir, however, for no special interest would attach to it unless HP were, unlike Basic Law V, consistent with second-order logic, and Parsons did not so much as raise the question whether it is. Almost twenty years later, however, Crispin Wright re-discovered what Parsons had observed and showed in detail how the axioms of arithmetic can be derived from HP in second-order logic (Wright, 1983). He also showed that an attempt to replicate Russell's paradox within the new system fails, and he conjectured that the new theory was in fact consistent. Once formulated, the conjecture was quickly proved.¹¹

¹¹ This observation was made, independently, by George Boolos, John Burgess, Allen Hazen, and

Call the second-order theory whose sole ‘non-logical’ axiom is HP *Frege Arithmetic*. Then it can be shown that Frege Arithmetic is consistent if second-order arithmetic is; that is, if Frege Arithmetic is inconsistent, so is second-order arithmetic. But if second-order arithmetic were ever shown to be inconsistent, that would precipitate a crisis in the foundations of mathematics that would make the discovery of Russell’s paradox look trivial by comparison. So Frege Arithmetic is (almost certainly) consistent. And second-order arithmetic can be interpreted within it: Given appropriate definitions, one really can prove the axioms of arithmetic from HP, in second-order logic. Or again: All truths of arithmetic follow logically from the principle—seemingly obvious, once one understands it—that the number of *F*s is the same as the number of *G*s if and only if the *F*s are in one-one correspondence with the *G*s. It is this surprising and beautiful result that is now called *Frege’s Theorem*. And study of the detailed, formal proof Frege gives of the axioms of arithmetic in *Grundgesetze* has shown that, charitably read, it does indeed amount to a proof of Frege’s Theorem (Heck, 2011b).

How is Frege’s Theorem proven? It would be inappropriate to give a complete proof of it here, but it is worth giving the reader some sense of how the proof goes. Let us write ‘ $Nx : Fx$ ’ to mean ‘the number of *x* such that *Fx*’; let ‘ $\text{Eq}_x(Fx, Gx)$ ’ abbreviate a formula that defines ‘the *F*s are equinumerous with the *G*s’.¹² Then HP can be written:

$$Nx : Fx = Nx : Gx \equiv \text{Eq}_x(Fx, Gx)$$

Frege then defines zero as the number of the concept non-self-identical:

$$0 = Nx : x \neq x$$

We also need to define ‘increase by one’. What Frege actually defines is a relation between numbers which we may call the relation of predecession. Intuitively, a number *m* is one less than a number *n* if a concept that has the number *m* is one object short of being a concept that has the number *n*. That is, *m* (immediately) precedes *n* just in case there is a concept *F* and an object *y* such that: the number of *F*s is *n*; *y* is an *F*; and the number of *F*s other than *y* is *m*. (Compare the definition of ‘*n* + 1’ considered on page 8.) Formally, Frege gives the definition:

$$Pmn \equiv \exists F \exists y [n = Nx : Fx \wedge Fy \wedge m = Nx : (Fx \wedge x \neq y)]$$

Harold Hodes.

¹² One of them is:

$$\begin{aligned} &\exists R [\forall x \forall y \forall z \forall w (Rxy \wedge Rzw \rightarrow x = z \equiv y = w) \wedge \\ &\quad \forall x (A(x) \rightarrow \exists y (Rxy \wedge B(y))) \wedge \\ &\quad \forall y (B(y) \rightarrow \exists x (Rxy \wedge A(x)))] \end{aligned}$$

But any of them will do.

Three of the usual axioms of arithmetic are now easily proven: that zero has no predecessor; that no number has more than one predecessor; and that no number has more than one successor.

Consider the first, for example. Suppose that zero did have a predecessor, that is, that, for some m , we had $Pm0$. Then, by definition, there would be a concept F and an object y such that:

$$0 = \text{Nx} : Fx \wedge Fy \wedge m = \text{Nx} : (Fx \wedge x \neq y)$$

That is to say, 0 is the number of F s, y is an F , and m is the number of F s other than y . *A fortiori*, there is a concept F whose number is 0 and under which some object falls. But that is impossible. For 0 is the number of the concept *non-self-identical* and so, if 0 is the number of F s, the number of F s is the same as the number of non-self-identical things; by HP, there must be a way of correlating the non-self-identical things one-to-one with the F s. But that there cannot be, if something, say b , is F : With which non-self-identical thing is b supposed to be correlated? So nothing precedes zero.

The proofs of the other two axioms mentioned are only a little harder.

There are two more axioms that need to be proven. These axioms make crucial reference to the notion of a natural, or finite, number, and that has not yet been defined. One of these axioms is the principle of arithmetical induction. Induction is a method for proving that all natural numbers have some particular property. The method is to show (i) that 0 has the property and (ii) that, if a given number n has it, then, its successor, $n + 1$ must also have it. Why does the method work? Well, suppose that both (i) and (ii) hold. Then, certainly, 0 has the property. And so, by (ii), $0 + 1$, i.e., 1, must also have it; hence, by (ii) again, $1 + 1$, i.e., 2 has it; so 3 has it; and so on. So, all natural numbers have it, because *all natural numbers can be 'reached' from 0 by adding one*.

Frege's definition of natural number in effect transforms the italicized clause into a definition. Here is something we know about the concept natural number: (i') 0 falls under it; and, (ii') whenever an object n falls under it, so does $n + 1$. Now, there are lots of concepts that satisfy conditions (i') and (ii'). For example, the concept *is self-identical* satisfies both of the conditions. (Exercise: Check this!) But not every concept satisfies the conditions: the concept *is identical with zero* does not, since 0 falls under the concept but $0 + 1$, i.e., 1, does not fall under it, so the concept *is identical with zero* fails to satisfy (ii'). So some concepts satisfy (i') and (ii') and some do not: call a concept *inductive* if it does. What can we say about the inductive concepts? Well, if a concept is inductive, surely every natural number must fall under it: 0 will, by (i'); so 1 will, by (ii'); and so on. Or, to put the point differently, if there is an inductive concept under which some object a does

not fall, then a is not a natural number. Conversely, if a is not a natural number, then there is an inductive concept under which it does not fall: namely, the concept *natural number* itself. So an object a is not a natural number if, and only if, there is an inductive concept under which a does not fall. Or, negating both sides of this claim:

An object a is a natural number if, and only if, a falls under every inductive concept.

And that, now, can be taken as a *definition* of the concept of a natural number. Note that it will all but fall out of the definition that proof by induction is valid: For if the hypotheses of the induction, (i) and (ii), are satisfied, then the property in question is inductive; so every natural number will have it.¹³

One might worry that there is some kind of circularity in the last paragraph. It is important to realize, however, that any circularity there might be does not lie in Frege's definition. We can say what an inductive concept *is* without appealing to the concept of a natural number. Formally, we have:

$$\text{Nat}(a) \equiv \forall F[F0 \wedge \forall x \forall y (Fx \wedge Pxy \rightarrow Fy) \rightarrow Fa]$$

Whatever circularity there might be lies, rather, in the argument used to motivate the definition, that is, in the argument given for the claim that *this definition properly defines the concept of a natural number*—and that sort of objection, as I said earlier, is in order here, since Frege needs his definitions to capture the meanings of the primitive arithmetical notions. This sort of objection was originally pressed by Henri Poincaré, but has more recently been developed by Charles Parsons (1983; 1995). Let me not discuss it further, however, except to say that it is bound up with the concerns about impredicativity mentioned earlier in connection with the comprehension axioms for second-order logic.¹⁴

The only axiom for arithmetic which we have not discussed is the most important: that every natural number has a successor. The axioms so far established do not, on their own, imply that there are infinitely many numbers; in fact, they are consistent with there being only one number, namely, zero. In their presence, however, the claim that every number has a successor implies that there are infinitely many numbers. For then 0 will have a successor, call it 1, from which it must be distinct; if 0 were 1, then 0 would precede itself, so 0 would have a predecessor, which it does not. But then 1 has a successor, call it 2; and 2 must be distinct

¹³ In fact, matters are somewhat more subtle than this makes them sound (Heck, 2014), but we need not consider those complications here.

¹⁴ I have discussed these arguments elsewhere (Heck, 2014) and argued that there is a way of defining the ancestral, different from Frege's, that avoids these sorts of objections.

from 0 (if not, 0 would again have a predecessor, namely, 1) and from 1 (otherwise 1 would precede itself and 0 would also precede it, contradicting the fact that no number has more than one predecessor). And then 2 will have a successor, call it 3, which will be distinct from 0, 1, and 2; and so on.

How, then, are we to prove that every number does have a successor? Frege's argument is too complex for me to explain it here, but the basic idea has already been mentioned above: We are to take the successor of 0 (that is, 1) to be $\text{Nx} : x = 0$; the successor of 1 (that is, 2), to be $\text{Nx} : (x = 0 \vee x = 1)$; and so forth. So, in general, the successor of a number n will be the number of x such that x is either 0 or 1 or ... or n .

This argument can be made rigorous, and it can be shown to work (Boolos, 1998a; Heck, 2011b; Boolos and Heck, 2011).

6 Closing: The Philosophical Significance of Frege's Theorem

So, Frege's Theorem shows that the axioms of arithmetic can be proven, in second-order logic, from HP. What are we to make of this fact philosophically? Does it show that logicism is true? Not, presumably, if logicism is the claim that the truths of arithmetic are truths of *logic*, for there is no good reason to suppose that HP is itself a truth of logic. Indeed, given how the notion of logical truth tends to be understood in contemporary philosophy, so that a truth of logic is something true in all interpretations, HP has just been proven *not* to be a truth of logic, since it is not true in any interpretation whose domain is finite.

Still, though, one might think that HP is, even if not a truth of *logic*, at least of a similar epistemological status. Wright (1983) suggests, for example, that it should be understood as embodying an 'explanation' of names of numbers or, to put the point less technically, that we can come to know that it is true simply by reflecting upon what number-words *mean*. If, in that sense, HP is analytic of the concept of number, then the axioms of arithmetic turn out to follow from a principle that is analytic in an extended sense, and that would seem enough to satisfy Frege's epistemological ambitions. If that is not logicism, surely it is rightful heir to the name.

This view has been the subject of a great deal of discussion since Wright proposed it. The main problems it faces may be divided into two groups: the Caesar problem and the 'bad company' objection. Enough has already been said about the former above: The Caesar objection, as was said, purports to show that HP *cannot* be taken as an explanation of names of numbers. Let me not discuss it further here.

The 'bad company' objection is so-called because it amounts to the charge that

HP is guilty of associating with the wrong sorts of principles. Even though HP is not itself inconsistent, it is of a form that can give rise to inconsistency. For example, Basic Law V is, as was said earlier, inconsistent, and the two principles are both of the form:

The blah of F is the same as the blah of G iff the F 's so-and-so the G 's.

Here 'so-and-so' stands for some relation between concepts which is, in technical parlance, an equivalence relation. (That is, it has the formal properties necessary to guarantee that the definition will not contradict the laws of identity.) So what's the problem? To quote a famous song, surely "One bad apple don't spoil the whole bunch of girls".¹⁵ But still, if there are problems that affect claims formally similar to a given one, then, even if they do not affect the given claim, it is natural to suspect that those problems will, upon closer examination, be seen to be manifestations of deeper problems that *do* affect the given one.

One way to develop this objection is as follows (Boolos, 1998b). The claim that HP is true is one that commits us, minimally, to its consistency; it is hard to see how we can claim to *know* HP to be true unless we are also prepared to claim that we *know* it to be consistent. Not that we do not know this: We do, as surely as we know much else we claim to know in mathematics. But the proof that it is consistent is one that can be formalized only within an exceptionally strong mathematical theory, one that (by Gödel's theorem) must be essentially stronger than Frege Arithmetic itself. And it seems to be asking us to swallow a lot to suppose that we can come to know anything like that simply by reflecting upon the concept of number.

This objection, however, is difficult to evaluate. For one thing, anyone attracted to the claim that arithmetic is analytic is already asking us to swallow a lot. The argument would seem to show, at best, that Wright ought to claim not just that we know HP to be true, but that we know it to be consistent, which is more. And what's a little more? Moreover, there are worries about the form of argument that was deployed in the last paragraph: It appears to be of a sort that drives many kinds of sceptical arguments. Compare: If I know that there is a computer on my desk, then I must also know that there is an external world. Is it supposed to follow that my claim to know that there is a computer on my desk depends upon my knowing *antecedently and independently* that there is an external world? If so, then we're in trouble, because it is hard to see how, if I suspend belief in all my particular items of knowledge about the world—i.e., subject that knowledge to Cartesian doubt—I

¹⁵ The song "One Bad Apple" was written by George Henry Jackson, a songwriter at the legendary Fame Studios in Muscle Shoals, Alabama—where I just happen to have been living at the time. It was recorded by The Osmonds, for whom it became a number one hit in the US.

could ever recover my senses. And it is a now familiar move in epistemology to deny that, just because some claim *A* implies some other claim *B*, a claim to know *A* must be *supported by* a claim to know *B*: We may well know *B* because we know *A*, not conversely. But then a similar move would appear to be available in this case: If we know that HP is true, then we know it is consistent; but it does not follow that our knowledge that it is true must rest upon independent knowledge of its consistency.

However good this response might be, though, there are other worries. If the view is supposed to be that every consistent principle of the general form mentioned above is 'analytic', it is refutable. Principles of that form are a dime a dozen. And some of the consistent ones are inconsistent *with each other*. Thus, for example, it is possible to write down principles of the same form that, though consistent, imply that there are only finitely many objects. Since HP implies that there are infinitely many objects, it is inconsistent with these ones. Worse, choose any proposition you like: It is possible to write down a principle of the form mentioned above that implies it; and, by the same token, it is possible to write down another that implies its negation.¹⁶ So not all consistent such principles can even be true, let alone analytically true. What is needed, then, is some way of distinguishing the 'good' principles from the 'bad' ones. But, while there has been a good deal of work on this problem, it still is not clear whether success is to be had. And even if it is, the question will remain whether there is any ground on which to claim that all the good principles are analytic.

It is worth saying, however, that one should not over-emphasize the importance of this question. Let it be granted that HP is not analytic. It might nonetheless be that it has a role to play in the foundation of our knowledge of the truths of arithmetic. HP does, after all, have powerful intuitive appeal, and the claim that it is, in some deep sense, integral to our understanding of the concept of number that concepts have the same number if, and only if, they are equinumerous could be true even if it is *not* true that HP is analytic, in any sense that will rescue Frege.¹⁷

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¹⁶ For the proof, see my paper "On the Consistency of Second-order Contextual Definitions" (Heck, 2011h).

¹⁷ See my paper "Counting, Cardinality, and Equinumerosity" (Heck, 2011a) for an attempt to defend this sort of suggestion.

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