Is Frege's Definition of the Ancestral Adequate?

Richard G Heck, Jr

Department of Philosophy, Brown University

1 Frege's Definitions of the Ancestral

Among the many things for which Frege's *Begriffsschrift* is celebrated, one of the most important is his definition of the ancestral. Frege first defines what it is for a concept[1] to be "herditary" in a "sequence", where a "sequence" is simply given by a relation. The definition is (Frege, 1967, §24):[2]

$$\mathrm{Her}_{xyz}(Fx; Ryz) \equiv \forall x \forall y(Fx \wedge Rxy \to Fy)$$

Frege then defines the strong[3] ancestral as (Frege 1967, §26; Frege, 1980, §79):

$$R^*ab \equiv \forall F(\mathrm{Her}_{xyz}(Fx; Rxy) \wedge \forall x(Rax \to Fx) \to Fb)$$

That is: a stands in the strong ancestral of R to b just in case b falls under every concept that is hereditary in the R-series and under which all immediate R-successors of a fall. The definition in *Grundgesetze* is the same, except that Frege defines the strong ancestral directly as:

$$R^*ab \equiv \forall F[\forall x \forall y(Fx \wedge Rxy \to Fy) \wedge \forall x(Rax \to Fx) \to Fb]$$

and does not bother with the intermediate definition of heredity (Frege, 2013, v. I, §45).

[1] Frege does not speak of concepts in *Begriffsschrift* but of predicates and functions, but these differences, as significant as they are (Heck and May, 2013), do not affect his defintion of the ancestral.

[2] I won't attempt to reproduce Frege's symbolism here, which differs between *Begriffsschrift* and *Grundgesetze*. Those differences are themselves of some interest (Heck, 201X, §4).

[3] So-called because we do not necessarily have R^*aa. Intuitively, we have R^*aa only when there is a "loop" in the R-series: a path from a back to itself.

The ancestral of course plays a crucial role in Frege's philosophy of arithmetic. In particular, the ancestral is used in the definition of the concept of a natural, or finite, number. Frege defines the weak ancestral as (Frege, 1967, §29; Frege, 1980, §81; Frege, 2013, v. I, §46):

$$R^{*=}ab \;\equiv\; R^{*}ab \vee a = b$$

Having then defined both the number 0 and what it is for one number a immediately to precede another number b in the "natural series of numbers",[4] which we shall write 'Pab', Frege then proceeds to define the concept of natural number as (Frege, 1980, §83; Frege, 2013, v. I, §46):[5]

$$\mathbb{N}a \;\equiv\; P^{*=}0a$$

It is an easy consequence of Theorems 128 and 144 of *Grundgesetze* that Frege could equivalently have defined the weak ancestral as:

$$R^{*=}ab \;\equiv\; \forall F(Fa \wedge \forall x \forall y(Fx \wedge Rxy \to Fy) \to Fb)$$

i.e., as: b falls under every R-hereditary concept under which a falls. So, the concept of natural number could equivalently have been defined as:

$$\mathbb{N}a \;\equiv\; \forall F[F0 \wedge \forall x \forall y(Fx \wedge Pxy \to Fy) \to Fa]$$

It is for this reason that it is so tempting to say, as Crispin Wright once did, that Frege's "account of the ancestral has made it possible... to *define* the natural numbers as entities for which induction holds..." (Wright, 1983, p. 161, his emphasis).

That is not quite right.[6] It is possible, by a series of elementary logical manipulations, to derive:

(IND-) $$\forall F[F0 \wedge \forall x \forall y(Fx \wedge Pxy \to Fy) \to \forall x(\mathbb{N}x \to Fx)]$$

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[4]The definitions are:

$$0 = \mathrm{N}x : x \neq x$$
$$Pab \;\equiv\; \exists F \exists y[b = \mathrm{N}x : Fx \wedge Fy \wedge a = \mathrm{N}x : (Fx \wedge x \neq y)]$$

where '$\mathrm{N}x : Fx$' means: the number of Fs. (In *Grundgesetze*, though not in *Die Grundlagen*, Frege actually defines P as the extension of a relation, but it is now customary to ignore this aspect of Frege's presentation.)

[5]Frege does not use any special symbol for this concept, but simply uses (his version of) "$P^{*=}0\xi$".

[6]I do not mean to pick on Wright here. This opinion seems to have been, and still to

from Frege's definition. But arithmetical induction is:

(IND) $\forall F[F0 \wedge \forall x \forall y(\mathbb{N}x \wedge Fx \wedge Pxy \to Fy) \to \forall x(\mathbb{N}x \to Fx)]$

which is a stronger principle, since the the second conjunct of the antecedent has been weakened by the addition of the condition '$\mathbb{N}x$'.[7] It might seem surprising that (IND-) and (IND) should be as different as they are. But in the present context, they are very different. Suppose, for example, that we work in a version of Frege's own theory in *Grundgesetze*, but take our background logic to be predicative second-order logic. This theory, as is now well-known, is consistent. And, if we define number exactly as Frege does, and use exactly his definitions of zero, predecession, and natural number, we can prove all the axioms of the weak arithmetical theory known as Robinson arithmetic (Heck, 1996). Moreover, we can prove (IND-), since, as mentioned, it follows trivially from the definition of natural number. But we cannot prove (IND).

Frege himself seems to have been insufficiently aware of the difference between these two principles in *Die Grundlagen*. His proof, in §§82–3, that every number has a successor purports to use only (IND-), whereas it is (IND) that is needed (Boolos and Heck, 2011, pp. 327ff). In the corresponding portions of *Grundgesetze*, however, Frege explains this difference himself (Frege, 2013, v. I, §114) and shows how (IND-) can be used to prove (IND). More precisely, he uses Theorem 144:

(Gg 144) $R^{*=}ab \wedge Fa \wedge \forall x \forall y(Fx \wedge Rxy \to Fy) \to Fb$

which is all but immediate from the definition of the weak ancestral, to prove Theorem 152:

(Gg 152) $R^{*=}ab \wedge Fa \wedge \forall x \forall y(R^{*=}ax \wedge Fx \wedge Rxy \to Fy) \to Fb$

of which arithmetical induction is then an instance: Just substitute $P\xi\eta$ for $R\xi\eta$ and 0 for a.

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be, fairly common, and it is something I thought myself for a long time.

[7]This point is especially clear when we think of the mathematical part of Frege's project in terms of the notion of *relative interpretation* (Tarski et al., 1953). Suppose our goal is to interpret first-order Dedekind-Peano arithmetic in some form of Frege arithmetic. Then, among the principles we need to interpret is the induction scheme

$$A(0) \wedge \forall x(A(x) \to A(Sx)) \to \forall x(A(x))$$

The interpretation will have to be relativized to the 'natural numbers', and that will then introduce the restriction $\mathbb{N}x$ in the antecedent of the second conjunct.

The proof of (Gg 152) is not difficult: We need only take $F\xi$ in (Gg 144) to be $R^{*=}a\xi \wedge F\xi$. That yields

$$R^{*=}ab \wedge (R^{*=}aa \wedge Fa) \wedge$$
$$\forall x \forall y[(R^{*=}ax \wedge Fx) \wedge Rxy \rightarrow (R^{*=}ay \wedge Fy)] \rightarrow Fb$$

Since the conjunct $R^{*=}aa$ is trivial, we thus have:

$$R^{*=}ab \wedge Fa \wedge$$
$$\forall x \forall y[(R^{*=}ax \wedge Fx) \wedge Rxy \rightarrow (R^{*=}ay \wedge Fy)] \rightarrow Fb$$

To prove (Gg 152), then, we need only show that

$$\forall x \forall y[(R^{*=}ax \wedge Fx) \wedge Rxy \rightarrow (R^{*=}ay \wedge Fy)]$$

follows from

$$\forall x \forall y(R^{*=}ax \wedge Fx \wedge Rxy \rightarrow Fy)$$

But if $R^{*=}ax \wedge Rxy$, then certainly $R^{*=}ay$, by a weak form of transitivity that Frege proves as Theorem 133.

Although this proof is easy, it is still worth noting that it requires non-trivial logical resources. In particular, it requires Π^1_1 comprehension. Frege does not have comprehension axioms in his system but, rather, a rule of substitution that allows, for example, the substitution of $R^{*=}a\xi \wedge F\xi$ for $F\xi$ in (Gg 144). But this is just equivalent to assuming comprehension for $R^{*=}a\xi \wedge F\xi$. And since the definition of $R^{*=}$ is Π^1_1—it is of the form $\forall F\phi$, where ϕ contains no second-order quantifiers—the crucial formula $R^{*=}a\xi \wedge F\xi$ is also Π^1_1. That is why (Gg 152), and so arithmetical induction, cannot be proven in the predicative fragment of the formal theory of *Grundgesetze* mentioned earlier. The lesson is thus that Frege does *not* simply define the natural numbers as the numbers for which arithmetical induction holds. He defines them as the numbers for which a weaker sort of induction-like principle holds, and then proves arithmetical induction from the weaker principle.

Despite these complications, however, Frege's definition of the ancestral, and of the concept of natural number, are adequate for any mathematical application one might need to make of them, as anyone familiar with Frege's arguments in *Grundgesetze* will know.[8] So, in that sense, Frege's definition of the ancestral undeniably works.

[8]Assuming, again, that we have enough comprehension.

But the definition can seem almost magical. One might well want to ask, with Wright (1983, p. 159), whether Frege's definition "capture[s] the intuitive meaning" of the ancestral. The question matters for several reasons, not least of which is that Frege is attempting to show that *arithmetic* is logical in character, and by that I mean arithmetic as we ordinarily understand it. If so, then Frege really does need a *definition* of the concept of natural number that "capture[s] the intuitive meaning" of that concept, not just one that works, in some technical sense.

What is the intuitive meaning we need to capture? Frege is aiming to tell us how to define the notion of an ancestor, for example, in terms of that of a parent. And there is an obvious way to do this: My ancestors are my parents, and their parents, and *their* parents, and so on. Of course, that uses "and so on", which is not very helpful. So we might try: My ancestors are those people to whom I can be connected by a finite series of steps, starting with me and moving always from a person to one of that person's parents. But that definition uses the notion of finitude and so could not be employed in a non-circular definition of the concept of natural number. That is what forces Frege to give a very different sort of definition. But that, in turn, raises the question what Frege's definition has to do with the ordinary notion of an ancestor. And the truth is that it is far from obvious that Frege's definition is even *extensionally* correct: that my 'Frege-ancestors' are exactly my ancestors. It is, that is to say, far from obvious that the people who have every property my parents have and that is had by any parent of someone who has it, are exactly the people to whom I can be connected by a finite series of steps, starting with me and moving always from a person to one of that person's parents. Couldn't some person totally unrelated to me just *happen* to have every such property (cf. Wright, 1983, p. 160)? Then Frege's definiendum would be satisfied, and the person in question would wrongly be counted as one of my ancestors.

In fact, however, the extensional correctness of Frege's definition can be proven. The intuitive notion we are trying to capture is that of one object's being some finite number of R-steps 'downstream' from another. So what we want to show is simply that, for any a, b, and R, if b is finitely many R-steps downstream from a, then R^*ab, and conversely.

Let us first prove the stated direction. Suppose that b is finitely many R-steps downstream from a. To show that R^*ab, we prove the formula that defines it:

(1) $$\forall F[\forall x(Rax \to Fx) \land \forall x \forall y(Fx \land Rxy \to Fy) \to Fb]$$

The proof is by arithmetical induction on how many R-steps b is from a. For the basis, suppose that b is one R-step downstream from a, i.e., that Rab. Fix F and suppose that $\forall x(Rax \rightarrow Fx)$. (We do not need the other conjunct of the antecedent.) Then certainly Fb, since Rab. For the induction step, then, suppose that (1) holds whenever b is n R-steps downstream from a, and suppose that c is $n+1$ R-steps downstream. Fix F and suppose that $\forall x \forall y(Fx \wedge Rxy \rightarrow Fy)$. (Again, we do not need the other conjunct of the antecedent.) Then there must be a b such that Rbc, where b is n R-steps downstream from a. But then we have Fb, by the induction hypothesis, and so Fc, since F is R-hereditary. So, if a is any finite number of R-steps downstream from b, then (1) holds.

Note, for later reference, that the proof just given used arithmetical induction.

For the proof of the converse, abbreviate "ξ is finitely many R-steps downstream from a" as: $\mathbf{FRS}(\xi)$. If Rax, then certainly $\mathbf{FRS}(x)$, since then x is precisely one R-step downstream from a. Similarly, if $\mathbf{FRS}(x)$ and Rxy, then also $\mathbf{FRS}(y)$, since, if x is n R-steps downstream, then y is $n+1$ steps downstream. But now we have shown that:

$$\forall x(Rax \rightarrow \mathbf{FRS}(x)) \wedge \forall x \forall y(\mathbf{FRS}(x) \wedge Rxy \rightarrow \mathbf{FRS}(y))$$

And so the definition of the ancestral delivers:[9]

$$R^*ab \rightarrow \mathbf{FRS}(b)$$

which is what we wanted to prove.

Note, for later reference, that this proof did *not* use arithmetical induction but only the sort of induction derivable from Frege's definition of the ancestral.

Frege's definition of the ancestral is thus extensionally correct. We can prove

(NEC*) If b is finitely many R-steps from a, then R^*ab

with the help of ordinary arithmetical induction. And we can prove

[9]The definition immediately yields:

$$R^*ab \rightarrow \forall F[\forall x \forall y(Fx \wedge Rxy \rightarrow Fy) \wedge \forall x(Rax \rightarrow Fx) \rightarrow Fb]$$

Now instantiate $F\xi$ with $\mathbf{FRS}(\xi)$. That gives:

$$R^*ab \rightarrow [\forall x(Rax \rightarrow \mathbf{FRS}(x)) \wedge \forall x \forall y(\mathbf{FRS}(x) \wedge Rxy \rightarrow \mathbf{FRS}(y)) \rightarrow \mathbf{FRS}(b)]$$

But the antecedent of the embedded conditional has been proven.

(SUF*) If R^*ab, then b is finitely many R-steps from a

using a different sort of induction.

2 Objections To Frege's Definition

If the foregoing arguments strike the reader as somehow circular, then they are not alone. Many such charges have been made over the years.

2.1 Poincaré

Henri Poincaré famously lodged several objections of circularity against attempts to define the concept of natural number in such a way as to render induction provable. Poincaré's own discussions are, to my mind, confusing at best, but they have been carefully elaborated by Janet Folina, who distinguishes four sorts of objection that Poincaré offers. I will follow Folina's presentation.[10]

The first objection is that, if one simply regards induction as part of the definition of natural number, then one needs a proof of the consistency of such a definition, and induction will figure essentially in that proof (Folina, 2006, pp. 278–9). But, as Folina notes, this objection rests upon a conflation between logicism and a sort of formalism that supposes that arithmetical notions are defined by the axioms in which they figure. That makes it particularly inappropriate as an objection to Frege, since he makes this sort of objection to formalism himself (Frege, 2013, v. II, §§138–45). Moreover, Frege is extremely hostile, at least in his mature period, to the idea that axioms can define anything (Frege, 1984b,c).

The fourth objection—we'll consider the second and third shortly—is that the 'new logic' of Frege and Russell is so different from traditional logic that it is not clear why it should be regarded as logic at all. Poincaré, as Folina reads him, thinks the logicist might seek to answer this question by appealing to the idea that the axioms and rules of inference governing the logical notions are analytic of them because they serve to define them. Then we need a consistency proof again (Folina, 2006, p. 280–1). But that makes the fourth objection a version of the first,

[10]The first and fourth objection seem to be discussed only in Folina's 2006 paper "Poincaré's Circularity Arguments for Mathematical Intuition". The second and third are also discussed in her 1992 book *Poincaré and the Philosophy of Mathematics*. As one might expect, the later discussion is better, but there are points that emerge only in the earlier one, so the reader is encouraged to consult both.

which, as I have said, is certainly not properly brought against Frege. Nonetheless, the general question Poincaré raises is perfectly reasonable, and neither Frege nor Russell says much explicitly about why the 'new logic' is properly so called.[11] The question is particularly pressing in the present setting, since the definition of the ancestral is given in second-order logic. More importantly, as we saw above, the proof of arithmetical induction from the definition of the ancestral requires Π^1_1 comprehension, and *impredicative* second-order logic is particularly liable to seem like "set theory in sheep's clothing", as Quine (1986, pp. 66–8) famously put it. But this sort of worry is not one about circularity.

Indeed, Poincaré's concerns about the logical character of the 'new logic' apply every bit as much to its first-order fragment as to anything higher-order. Those concerns emerge in the second and third objections that Folina (2006, pp. 279–80) distinguishes, which I'll present together, since they seem to me to be closely related. The basic objection is that mathematics must be used in the very "Exposition of the Concept-script", as Frege entitles Part I of *Grundgesetze*. Specifically, various sorts of recursive definition must be employed, for example, in explaining what a well-formed formula is, or what a proof is (Parsons, 1995, p. 202).

Folina goes so far as to attribute to Poincaré the claim that "induction. . . is epistemologically prior, not only to arithmetic, but to any sort of systematic thinking" (Folina, 1992, p. 103). That has to be too strong. Even if we allow that it is essential for us to think in symbols, it does not follow that we need to be able to apprehend the symbols as symbols, let alone that we need to know anything about them. In brief: We do not need to think *about* symbols to think *with* symbols. So, even if a proper understanding of the symbols themselves, and of the relations between them that underlie logical inference, would require the use of inductive methods, there is no reason that *we*, the thinkers, have to use such inductive methods in order to make logical inferences and to be justified in drawing the conclusions at which we thereby arrive. Note that I am *not* saying that it is an uninteresting question whether, as a matter of empirical fact, our initial appreciation of recursion somehow flows from our understanding of language. Maybe it does. And, if it does, then I would suppose that fact to be of significant psychological interest. But that is where its interest would lie: in psychology, not in epistemology.

[11] As it happens, I think Frege did have some inchoate views about the matter (Heck, 2010; 2012, ch. 2). We'll return to this issue.

This sort of charge, that Poincaré's argument is unacceptably psychologistic, has been made before, by Warren Goldfarb, but it is worth being clear exactly what the response is and what its presuppositions are. The first charge Goldfarb (1988, p. 68) makes is that Poincaré ignores the distinction "between empirical conditions a person must satisfy in order to arrive at certain propositions and the ultimate rational basis for the propositions". Ironically, Goldfarb claims, this distinction is itself supposed to be founded on the 'new logic', so Poincaré is unable to appreciate it precisely becuase he rejects the 'new logic'. But, while it is certainly true that Frege and Russell intended the 'new logic' to be a tool for investigating epistemic relations, I do not see any reason to suppose that one's ability to draw the distinction between enabling conditions and justificatory relations depends upon whether one is with Aristotle and Boole or with Frege and Russell.

That, however, is not really the crucial point. Poincaré has a specific reason to think that induction plays a justifying role in logical inference. Folina suggests, in fact, that "Poincaré's most pentrating criticism" is that recursion is implicit in the inferential rules of the 'new logic' (Folina, 2006, p. 280; see also 1992, pp. 84–9). To determine whether a formula is an instance of modus ponens, for example, we would need to be able to count parentheses (or to do something equivalent), and the problem only gets worse with the axioms and rules involving the quantifiers. But that looks like a recursive process,[12] and recursion is the flip-side of induction.

Goldfarb insists, however, that this argument, too, is overly psychologistic:

> Logic is not about manipulations of signs on paper, even though it may be a psychological necessity for us, in order to be sure that we are proceeding logically, to verify proofs by syntactic means. (Goldfarb, 1988, p. 69)

And that seems right. The 'new logic' is supposed to make it possible for us to know *that* my beliefs that $A \to B$ and A suffice to justify my belief that B by observing that a certain syntactic relation holds among these three propositions—and, ultimately, to know *that* the Basic Laws of *Grundgesetze* suffice to justify its theorems. But that simply does not imply that, if I infer B from $A \to B$ and A, then my justification for B

[12] Can it be proven to be a recursive process? Might one think of that as a consequence of the fact that no regular expression can be used to check for balanced parentheses?

includes the meta-judgement that such a syntactic relationship obtains. On the contrary, as was just said, my justification for B is simply that $A \to B$ and A.

In leading up to this response, however, Goldfarb seems to imply that its availability depends upon adoption of the so-called 'universalist' conception of logic that he and others have claimed to find in Frege[13] or, perhaps more precisely, upon the claim that "no metatheoretical stance [on logic is] either available or needed" (Goldfarb, 1988, p. 69). But even if one does think such a stance is possible, one can continue to insist, with Goldfarb's logicist, that "To give the ultimate basis for a proposition is... to assert the proposition with its ground, not to assert the metaproposition 'this sentence is a theorem'" (Goldfarb, 1988, p. 69). The only thing that would prevent one from doing so would be the view that the metaproposition was epistemically more fundamental than the proposition itself. But it is hard to imagine any sensible view that would endorse that claim, and the mere fact that a "metatheoretical stance" is available cannot by itself imply that it plays the sort of epistemic role that Poincaré supposes it must.

And that is a very good thing. Poincaré's question whether the 'new logic' really should be so called is, as I said above, a perfectly good one. But, as Folina (2006, p. 285) emphasizes, the commitments Goldfarb ascribes to Frege threaten to "render all questions about the general nature of logic unanswerable in a noncircular way by logic...", with the further consequence that logicism would be "unable to be justified in a way that would be approved by the program itself". And, indeed, those who regard Frege as a universalist in Goldfarb's sense (see e.g., Ricketts, 1997, p. 174) frequently emphasize that such a doctrine would preclude Frege from making genuine sense of the question: What distinguishes logical from non-logical truth? If so, however, Frege would have to deny that it is even a substantive question whether, say, Basic Law V is a truth of logic (prescinding, for the moment, from its inconsistency), or whether impredicative second-order reasoning is properly logical, in which case logicism threatens to become a merely verbal doctrine (Heck, 2012, p. 34).[14]

[13]There is now a sizeable literature on this. Proponents include Goldfarb (2001), Kremer (2000), Ricketts (1986), and Weiner (1990). Opponents include Stanley (1996), Sullivan (2005), Tappenden (1997), and me (Heck, 2010; 2012, Part I).

[14]Special thanks to an anonymous referee for helping me clean up the discussion of Goldfarb.

2.2 Kerry and Angelelli

It seems to me, therefore, that good answers to Poincaré's objections of circularity are available to Frege. But there are more such objections to be considered. Ignacio Angelelli discusses two in his paper "Frege's Ancestral and Its Circularities". The second of these is originally due to Benno Kerry, who raises it in the same series of articles in which he introduces the infamous problem of the concept *horse*. Speaking of Frege's definition of the ancestral, Kerry writes:

> Now, this criterion is to begin with of dubious value because there is not a catalogue of such properties [the hereditary ones, etc], hence one is never sure that one has examined the totality of them. Moreover, there is the crucial fact that, as [Frege] has proved, of the properties that are hereditary in the f-series is also the following: to follow x in the f-series. Thus, the determination of whether y follows x in the f-series, according to the definition given for this concept, depends on whether, in addition to a lot of other things about hereditary properties in general, one knows, in particular, about the hereditary property "being a descendant of x", that y has it or not. It is clear that this circle should totally prevent us from saying, in Frege's sense, that any y follows x in an f-series. (Kerry, 1887, p. 295, as translated in Angelelli, 2012, p. 480)

Kerry's worry is this. Frege's definition tells us that y follows x in the f-series just in case y has every f-hereditary property that x's immediate f-successors all have. But the property *follows x in the f-series* is precisely such a property. So, to determine whether y has every such property, we need to determine whether y has this property, that is, we need to determine whether y follows x in the f-series. But that is what we were trying to determine in the first place.

There is not really any circularity here, however, for reasons Russell makes clear in the appendix on Frege in the *Principles*:

> This argument... radically misconceives the nature of deduction. In deduction, a proposition is proved to hold concerning *every* member of a class, and may then be asserted of a particular member: but the proposition concerning *every* does not necessarily result from enumeration of the entries in a catalogue. (Russell, 1903, p. 522)

Russell presents the point as a logical one: A proof that $\forall F(\ldots F \ldots)$ need not consist in a proof of each of its instances. But, as so often with Russell, the fundamental point is really epistemological: Kerry's objection assumes that defense of the universal claim $\forall F(\ldots F \ldots)$ must depend upon our having some *independent* justification for each of its instances. But that is simply false. We can have justification for the universal claim and thereby have justification for its instances.

The same sort of answer can be given to the other charge of circularity that Angelelli (2012, pp. 478–9) brings against Frege's definition. Angelelli has us imagine that, if Fritz can show that Karl is his ancestor, he stands to inherit a large sum of money. So, attempting to avoid the need to show that Karl is his ancestor in the *ordinary* sense, he asserts that Karl is his ancestor in Frege's sense.

> Fritz does not fully understand what he is saying but hopes that the audience will be intimidated by such a display of conceptual weaponry.... Alas, a smart opponent ruins this plan by requesting Fritz to defend the Fregean version of his claim for the predicate "Karl is an ordinary ancestor of x". Fritz is dialogically forced to assert the conditional: if (the property "Karl is an ordinary ancestor of x" is hereditary and all children of Karl have it) then Karl is an ordinary ancestor of Fritz. The opponent then attacks the conditional by asserting its antecedent, which is a conjunction whose two conjuncts the opponent defends successfully. Thus, poor Fritz is left with the obligation of defending the same statement he wanted to shun: "Karl is an ordinary ancestor of Fritz". (Angelelli, 2012, p. 479)

But this would have no force whatsoever if Fritz had *grounds* for the universal generalization. If he had such grounds, then, given the conditional and his opponent's helpful proof of its antecedent, he could perform a simple inference to get the result he needs. In any event, it is not clear what conclusions we can draw from cases in which people make assertions they do not even understand.

One might respond that Angelelli (2012, pp. 479–80) is specifically concerned to argue that this sort of circularity undermines the status of Frege's definition of the ancestral as any kind of *analysis*. He mentions Russell's answer to Kerry himself, for example, but then raises the question whether it depends upon assumptions about how the definition is intended. But it seems pretty clear that it does not. Russell is making an elementary logical *cum* epistemological point: A universal generalization does not have to be derived from the conjunction of its instances, and knowledge of a universal generalization need not rest upon knowledge of its instances.

What really seems to be bothering Kerry, and Angelelli, too, is whether, say, the fact that Karl is Fritz's Frege-ancestor could be established otherwise than by exhibiting a parental path connecting them. If not, the thought seems to be, then Frege's definition essentially depends upon the intuitive one. In fact, however, this is not at all clear: Even if knowledge that Karl is Fritz's Frege-ancestor depends upon knowledge of the existence of a path, the bearing of this *epistemic* dependence upon the correctness of Frege's definition is not clear. And it simply isn't true that one can only establish that Karl is Fritz's Frege-ancestor by exhibiting such a path. Suppose Fritz knew by testimony that Karl's father was his Frege-ancestor, and suppose he also knew that Karl was an only child. Then he could prove that Karl was his Frege-ancestor in roughly the way Frege proves Theorem 124 of *Begriffsschrift*.[15] Whether showing that Karl is Fritz's Frege-ancestor requires us to exhibit a parental path therefore depends, unsurprisingly, upon what else we happen to know. Moreover, the power of Frege's definition shows itself not in particular cases but in results like the one just mentioned: in the *generalizations* about the ancestral that it allows us to prove. Worries about what happens in particular cases thus seem to be misplaced.

2.3 Papert and Parsons

There is, however, a deeper worry about Frege's definition of the ancestral, one that Charles Parsons (1995, pp. 203–5) discusses in his paper "Frege's Theory of Number", attributing it to Seymour Papert (1960).[16] The worry is that the argument for the extensional adequacy of Frege's definition of the ancestral takes the intuitive notion of finitude for granted and assumes the correctness of ordinary induction. It is

[15] In our notation, this is:

$$\forall x \forall y \forall z (Rxz \wedge Ryz \rightarrow x = y) \wedge R^{*}ac \wedge Rbc \rightarrow R^{*=}ab$$

We do not really need to know, however, that R is one-many, only that b is the unique R-predecessor of c.

[16] Unfortunately, I do not read French well enough to confirm this attribution. But Parsons is nothing if not careful, so I'm prepared to take him at his word.

easy to see how this observation might lead one to worry that Frege's definition of the ancestral cannot provide us with any sort of *justification* for ordinary arithmetical induction, even though it does allow us to prove something that looks very much like it. The problem is that we have to use arithmetical induction to convince ourselves of the correctness of the definition we use to prove arithmetical induction. And the more general worry is that this same sort of situation will arise for every definition of the ancestral.[17]

To spell this out a bit, Parsons and Papert are conceding that Frege can show that

(ANC) $$\forall F[F0 \wedge \forall x \forall y(P^{*=}0x \wedge Fx \wedge Pxy \rightarrow Fy) \rightarrow \forall x(P^{*=}0x \rightarrow Fx)]$$

is a truth of logic. What they are challenging is the claim that this shows that

(IND) $$\forall F[F0 \wedge \forall x \forall y(\mathbb{N}x \wedge Fx \wedge Pxy \rightarrow Fy) \rightarrow \forall x(\mathbb{N}x \rightarrow Fx)]$$

is a truth of logic. Now, we can of course derive (IND) from (ANC) and

(EQ) $$\forall x(P^{*=}0x \equiv \mathbb{N}x)$$

which will license us to replace the two occurences of $P^{*=}0x$ with $\mathbb{N}x$ in (IND). So (IND) is a truth of logic if (EQ) is. But note that we do need both directions of (EQ): We need the left-to-right direction to replace the first occurrence of $P^{*=}0x$ and the right-to-left direction to replace the second one. But the right-to-left direction of (EQ) is a special case of NEC*, whose proof depends upon arithmetical induction, that is, upon

[17] It may well be that the Frege of *Grundgesetze* would have no time at all for this sort of objection. But the Frege of *Die Grundlagen* might be committed to taking it seriously. Thus, in explaining his notion of analyticity, Frege writes:

> The problem becomes, in fact, that of finding the proof of the proposition, and of following it up right back to the primitive truths. If, in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one, bearing in mind that *we must take account also of all propositions upon which the admissibility of any of the definitions depends*. (Frege, 1980, §3, my emphasis)

To be sure, it's not entirely clear what Frege means here. But here's one thought. Suppose that, at least early on in the composition of *Die Grundlagen*, Frege was intending to use 'Hume's Principle' to introduce names of numbers, only later to change his mind in the face of the Caesar problem (Heck, 201X, §7). Then he might have meant, e.g., the proposition that equinumerosity is an equivalence relation, upon which the legitimacy

(IND).[18] So the worry is not that Frege's definition of the ancestral is circular, or that the definition depends upon arithmetical induction, or anything of that sort. The worry is that the appeal to arithmetical induction in the proof of (EQ) undermines Frege's claim to have *justified* arithmetical induction (let alone to have done so purely logically). The circularity, that is to say, is supposed to be epistemological, not logical.

Some will want to respond that Frege was not really in the business of analyzing the ordinary notion of the ancestral (or the related notion of finitude) but of replacing it with a rigourously defined notion suitable for the purposes of science. But that response threatens to divorce Frege's 'natural numbers' from the natural numbers, thus raising the question whether Frege really shows us how to prove axioms for arithmetic, not just something that look like axioms for arithmetic (Heck, 2011b, §1.3). And, in fact, we aren't even talking, at this point, about whether Frege's definition gets the *sense* of arithmetical claims (even close to) right. The issue here is purely extensional: If (EQ) is not true, then a statement of the form $\forall x(P^{*=}0x \rightarrow Fx)$ is not even about the same *objects* as a statement of the form $\forall x(\mathbb{N}x \rightarrow Fx)$. So the question is what reason we have to believe that Frege has managed to pick out the right objects, from among the cardinal numbers, as the finite ones.[19]

A different response is that this sort of circularity is familiar in other settings and is not necessarily problematic. For some years, there has been a robust discussion, for example, about the significance of the fact that modus ponens has to be applied in proving the validity of modus ponens.[20] Such a circularity would no doubt be fatal if we were in the business of trying to justify the laws of logic *ex nihilo*. But surely that is not what anyone supposes such arguments might accomplish since, as Dummett (1991, p. 204) puts it, "... there is no sceptic who denies the validity of all principles of deductive reasoning, and, if there were, there would obviously be no reasoning with him". Perhaps, then, we could think of the circularity affecting the justification of Frege's definition of the ancestral in a similar way.

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of that 'contextual definition' would depend.

[18]Alternatively, one can give a direct proof. Certainly $P^{*=}00$; and if $P^{*=}0x$ and Pxy, then $P^{*=}0y$. So *by arithmetical induction*, if $\mathbb{N}x$, then $P^{*=}0x$.

[19]And here again, the issue is not whether Frege's zero is the ordinary zero. Even if that is granted, the question in the text remains.

[20]The modern discussion probably begins with Dummett (1978). But if one is looking for a place to enter the current literature, one might start with Boghossian (2000) and Wright (2001).

Perhaps. And I would not want to discourage anyone from considering the issue in those terms. The best way to answer this sort of objection, however, or it seems to me, would be to give a definition of the ancestral that (i) does not appeal, either directly or indirectly, to the notion of finitude it seeks to explicate and (ii) has some claim to be *in*tensionally correct, so that its *ex*tensional correctness will not require proof at all, let alone proof by arithmetical induction.

Let's see if we can improve on Frege's definition, then.

3 A New Definition of the Ancestral

Consider an arbitrary relation R and some initial object a. We want to give a rigorous definition of the intuitive notion of (what I shall call) an R-descendant of a, where b is an R-descendant of a if b is reachable from a by a sequence of R-steps. That is, what we want to capture is the idea that there is an R-path from a to b. For technical reasons, the case where $a = b$ causes a lot of problems here, so it is easiest to define an analogue of the weak ancestral first: either $a = b$ or else $a \neq b$ and there is an R-path from a to b. The strong ancestral can then be defined in terms of the weak, as:

$$R_{*}ab \equiv \exists y(R_{*}^{=}ay \wedge Ryb)$$

where $R_{*}^{=}$ is the notion we are about to define.

An R-path from a to b would be described by a relation Q that restricted R, in the sense that $\forall x \forall y(Qxy \rightarrow Rxy)$, and that met some other conditions. Which? First, Q should be one-one—the steps back and forth along the path should always be completely determined—and the path described should begin with a and end with b: $\exists x(Qax)$ but $\neg\exists x(Qxa)$, and $\exists x(Qxb)$ but $\neg\exists x(Qbx)$. Moreover, everything else in Q's field—in the union of its domain and range—should be a 'step along the way':

$$\exists y(Qyx) \wedge x \neq b \rightarrow \exists z(Qxz)$$
$$\exists y(Qxy) \wedge x \neq a \rightarrow \exists z(Qzx)$$

That is to say: $\exists y(Qxy) \equiv \exists y(Qyx)$, except in the case of a and b. I'll put this by saying that Q 'runs from a to b' and abbreviate that claim as: $a \overset{Q}{\dashrightarrow} b$.

By themselves, these conditions do not suffice, as one can see from the following example. Let $R\xi\eta$ be the relation $\eta = \xi + 1$, defined on the integers. Let $Q\xi\eta$ be its restriction to integers ≥ 10 or ≤ -10. Then Q

satisfies the conditions just stated with $a = 10$ and $b = -10$, but -10 is certainly not reachable from 10 by a sequence of $+1$ steps. So the conditions so far stated are insufficient to capture the intuitive notion. This example also shows that it will not help to require that there should be no further restriction of Q meeting the same conditions. For that is true of Q in this example.

It is at this sort of point that Poincaré *et alia* might be expected to observe that the missing condition is that the field of Q should be finite. And, of course, there are many definitions of finitude to which we might appeal in attempting to state this additional condition. For example, we might appeal to a definition due to Zermelo: A set is finite if it can be well-ordered by a relation whose converse is also a well-ordering.[21] Zermelo's definition has no claim, however, to capture the intutive notion of finitude, so a proof that it was even extensionally equivalent to it would then be wanted. Such a proof would likely use induction. Even if did not, however, our definition of an R-descendant would contribute little to the analysis of finitude, since it would actually depend upon a prior analysis of that very notion.

Familiarly, however, there are two notions of finitude. Perhaps the more familiar of these is the one we find in Cantor: one that is connected with such notions as enumeration, recursion, and induction. That is the notion we are presently trying to analyze. But there is a different notion that was introduced by Dedekind (1902, §64): A set is Dedekind finite if it is not equinumerous with any of its proper subsets. That is *not* the notion we are trying to analyze—it is defined as Dedekind defined it—so it would be perfectly in order for us to appeal to the notion of *Dedekind* finitude in our analysis of the *enumerative* notion of finitude. And it turns out that it is sufficient to require that the field of Q be Dedekind finite or, more simply: The field of Q must not be equinumerous with the result of omitting a.[22]

It might seem surprising that this would suffice, since, without an

[21] Parsons (1987) discusses the history of such definitions.

[22] Antonelli (2012) has made a closely related point, as did Albert Visser in a lecture given in London in May 2012. So this idea seems to have been in the air. My own appreciation of it dates to at least 2004, when it appears in an early draft of what would eventually become *Reading Frege's Grundgesetze*.

The issue about the correctness of Frege's definition bothered me from the very beginning, and my initial attempts to address it, in the early 1990s, foundered upon the need to appeal to some notion of finitude in giving an adequate analysis of an R-path. It was reflection on the results mentioned in note 26 that made me realize that it would suffice to require that the path be Dedekind finite.

axiom of (countable) choice, one cannot prove that every Dedekind finite set is finite in the enumerative sense.[23] So one might have suspected that using Dedekind's notion here would lead us to a definition on which objects that are some infinite but *Dedekind* finite number of R-steps from a count as R-descendants of a. But it does not take very much thought to see, and we are about to prove, that no R-descendant of a can be infinitely many R-steps downstream from a.

The proposed definition of an R-descendant, then, is:[24]

$$\begin{aligned} R^{=}_{*}ab \equiv a = b \vee \exists Q[&\forall x \forall y(Qxy \rightarrow Rxy) \wedge \\ &\forall x\forall y\forall z(Qxy \wedge Qxz \rightarrow y = z) \wedge \\ &\forall x\forall y\forall z(Qxz \wedge Qyz \rightarrow x = y) \wedge \\ &\exists x(Qax) \wedge \neg\exists x(Qxa) \wedge \\ &\exists x(Qxb) \wedge \neg\exists x(Qbx) \wedge \\ &(\exists y(Qxy) \wedge x \neq a \rightarrow \exists z(Qzx)) \wedge \\ &(\exists y(Qyx) \wedge x \neq b \rightarrow \exists z(Qxz)) \wedge \\ &\neg \mathbf{Eq}_x(\exists y(Qxy \vee Qyx); \exists y(Qxy \vee Qyx) \wedge x \neq a)] \end{aligned}$$

or, abbreviating:

$$\begin{aligned} R^{=}_{*}ab \equiv a = b \vee \exists Q[&\forall x \forall y(Qxy \rightarrow Rxy) \wedge \\ &Q \text{ is one-one} \wedge \\ &a \overset{Q}{\dashrightarrow} b \wedge \\ &\neg \mathbf{Eq}_x(\exists y(Qxy \vee Qyx); \exists y(Qxy \vee Qyx) \wedge x \neq a)] \end{aligned}$$

where "$\mathbf{Eq}_x(Fx; Gx)$" means that F is equinumerous with G (that notion being defined in the usual way).[25]

In order to show that this definition could take the place of Frege's

[23]I discuss Dedekind's proof, and where choice is used in it, in *Reading Frege's Grundgesetze* (Heck, 2012, §11.3).

[24]As we shall see below, we do not need all of these conditions, but the analysis suggests them, and including extra conditions cannot hurt. Indeed, nothing here requires that Q be minimal, so, as the definition is formulated, the relation $Q01, Q\omega\omega$ would exhibit the fact that $P^{=}_{*}01$, since $P01$ and $P\omega\omega$. That's obviously not what we had in mind, so such a condition may well be worth adding to the analysis. The proofs then need minor modification.

[25]Note that a definition of finitude is implicit in the definition of $R^{=}_{*}$. The simplest

definition of the ancestral, it is enough to prove that it is equivalent to it:

$$R^{*=}ab \equiv R_*^{=}ab$$

The right-to-left direction follows from results Frege proves in *Grundgesetze*. The case where $a = b$ is of course trivial, so assume that $a \neq b$, and suppose that there is an R-path from a to b, i.e., a relation Q meeting the stated conditions. Suppose, for *reductio*, that $\neg Q^*ab$, and now consider the concept: $Q^{*=}a\xi$. Then we have:

(2) $$\forall x(Q^{*=}ax \to \forall y \forall z(Qxy \wedge Qxz \to y = z))$$

(3) $$\forall x(Q^{*=}ax \to \exists y(Qxy))$$

(4) $$\forall x(Q^{*=}ax \to \neg Qxx)$$

The first of these is immediate from the fact that Q is one-one. For the second, suppose $Q^{*=}ax$. If $x = a$, then $\exists y(Qxy)$, by the condition $\exists y(Qay)$, so suppose $x \neq a$. So Q^*ax. But then Theorem 124 of *Grundgesetze*:

(Gg 124) $$Q^*ax \to \exists y(Qyx)$$

implies that $\exists y(Qyx)$. And since $\neg Q^*ab$, $x \neq b$. So the condition

$$\exists y(Qyx) \wedge x \neq b \to \exists z(Qxz)$$

on Q yields (3). Finally, (4) is a essentially a version of Theorem 145 of *Grundgesetze*:

(Gg 145) $$P^{*=}0x \to \neg P^*xx$$

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version would be:

$$\begin{aligned}\text{Finite}(F) \equiv \exists Q \exists a \exists b[&\forall x(Fx \equiv \exists y(Qxy \vee Qyx)) \wedge \\ &Q \text{ is one-one} \wedge \\ &a \overset{Q}{\dashrightarrow} b \wedge \\ &\neg \mathbf{Eq}_x(Fx; Fx \wedge x \neq a)]\end{aligned}$$

That more or less mimics the way Frege characterizes finitude in terms of the ancestral (Heck, 2012, §8.1).

There are presumably historical antecedents for such a definition, but it still seems a bit surprising to me, anyway, that finitude can be defined in terms of Dedekind finitude. A somewhat similar definition would be: A set is finite if it is Dedekind finite and can be totally ordered. Proving that equivalence seems to require the axiom of choice, but it highlights the similarity between this defintion and the one due to Zermelo mentioned earlier.

which says that no natural number follows after itself in the natural series of numbers. The proof of (Gg 145) actually uses no more about P than that it is one-many and that $\neg P00$, however, and so actually establishes:

(5) $\forall x \forall y \forall z(Qxz \wedge Qyz \rightarrow x = y) \wedge \neg Q^{*}aa \rightarrow \forall x(Q^{*=}ax \rightarrow \neg Q^{*}xx)$

But the first conjunct is one of the conditions on Q, and the second again follows from (Gg 124), since, if $Q^{*}aa$, then $\exists y(Qya)$, which violates another of the conditions on Q.

As a little thought will show, conditions (2)–(4) characterize the Q-series beginning with a as an ω-sequence,[26] and Frege himself proves, essentially as Theorem 262 of *Grundgesetze*, that the number of objects in any series meeting these conditions is the same as the number of natural numbers and so is Dedekind infinite. Indeed, under these conditions, the relation Q itself correlates $Q^{*=}a\xi$ one-one with $Q^{*=}a\xi \wedge x \neq a$. But that contradicts the last condition on Q. So $Q^{*}ab$, whence also $R^{*}ab$, since Q restricts R.

For the other direction, suppose that $R^{*=}ab$. Then either $a = b$ or $R^{*}ab$. If the former, then of course $R_{*}^{=}ab$, trivially, so we need only prove that, if $R^{*}ab$, then $R_{*}^{=}ab$. We do so by induction. So we need to establish that:

$$\forall x(Rax \rightarrow R_{*}^{=}ax)$$

i.e., that every immediate R-descendant of a falls under the concept $R_{*}^{=}a\xi$, and that:

$$\forall x \forall y(R_{*}^{=}ax \wedge Rxy \rightarrow R_{*}^{=}ay)$$

i.e., that $R_{*}^{=}a\xi$ is hereditary in the R-series.

For the first, suppose that Rab. If $a = b$, then $R_{*}^{=}ab$ trivially. And if $a \neq b$, then the relation $\xi = a \wedge \zeta = b$ will describe an R-path from a to b, i.e., it will meet the conditions on Q in the definition of $R_{*}^{=}ab$.

[26]The basic reason we do not need choice here is really that we can prove, quite generally, that if Q is many-one, then $Q^{*=}a\xi$ is either finite or countably infinite. Frege proves a version of this fact as Theorems 207 and 263 of *Grundgesetze*. That same observation was what lay behind my proof of the result stated at the end of "*Die Grundlagen der Arithmetik* §§82–83" (Boolos and Heck, 2011, p. 85). A different way to put what is essentially the same point is that, if (2) and (4) hold, then $Q^{*=}a\xi$ is well-ordered by Q^{*}. (See Frege's proof of Theorem 359 in *Grundgesetze* and my discussion of that result (Heck, 2012, §§85, 9.2.1).) The conditions on an R-path—specifically, the condition that Q must be one-one—therefore imply that the path cannot be infinite and Dedekind finite.

For the second, suppose that $R^{=}_{*}ab$ and Rbc. If $a = c$, then again $R^{=}_{*}ac$ trivially, so suppose $a \neq c$. If $a = b$, then Rac, so $R^{=}_{*}ac$ by the argument just given. So suppose $a \neq b$. Then there is some Q meeting the conditions on $R^{=}_{*}ab$. We want to show that we can define Q' meeting the conditions for $R^{=}_{*}ac$. Now, if c is not in the field of Q, this is easy. Just adjoin $<b, c>$ to Q: $Q'\xi\zeta \equiv Q\xi\zeta \wedge (\xi = b \wedge \zeta = c)$. If c is in the field of Q, then there are two possibilities: Either $Q^{*}ac$ or $\neg Q^{*}ac$.[a]

If $Q^{*}ac$, then c is already on the path from a to b, and we can get the relation we need by restricting Q as follows: $Q'\xi\zeta \equiv Q\xi\zeta \wedge Q^{*}\xi c$. Most of the conditions then hold of Q' simply because they already held for Q. The only conditions that really need checking are the last one, concerning Dedekind finitude, and the condition that $\neg\exists z(Q'cz)$. But if $Q'cz$, then Qcz and $Q^{*}cc$, contradicting (5) above. And if the field of Q' were Dedekind infinite, then the field of Q would be, too.

If $\neg Q^{*}ac$, then c is not on the path from a to b. We can restrict Q to that path by considering $Q\xi\zeta \wedge Q^{*=}\zeta b$. Note that $\neg Q^{*=}cb$, and so $\neg\exists x(Qxc \wedge Q^{*=}cb)$. For, by what was shown in proving the right-to-left direction, $Q^{*=}ab$. So, if $Q^{*=}cb$, then, since Q is 1-1, we have either $Q^{*}ac$ or $a = c$ or $Q^{*}ca$, by a version of *Begriffsschrift* Theorem 133. But all of these are impossible, the last because $\neg\exists x(Qxa)$. Moreover, $\neg\exists x(Qbx \wedge Q^{*=}xb)$, since otherwise $Q^{*}bb$, contradicting (5), again. So we are free to adjoin $<b, c>$ not to $Q\xi\zeta$ but to $Q\xi\zeta \wedge Q^{*=}\zeta b$, and the relation we want can be defined as follows:

$$Q'\xi\zeta \equiv (Q\xi\zeta \wedge Q^{*=}\zeta b) \vee (\xi = b \wedge \zeta = c)$$

And now the conditions on Q' follow, once again, from the corresponding conditions on Q.

We have thus shown that $R^{*=}ab \equiv R^{=}_{*}ab$. And, though the proof of the left-to-right direction was by induction, that use of induction is justified directly by the definition of R^{*}, that is, by Frege's definition of the ancestral. At *no* point in the argument do we need to use *arithmetical* induction.

[a]Thanks for Ran Lanzet for pointing out a significant lacuna in the proof given in the published version of this paper: The case where $\neg Q^{*}ac$ was not considered. The next two paragraphs fix this.

I seem to recall that, in some earlier version, I had introduced a 'simplifying assumption' that everything in the field of Q was between a and b, i.e., that $Qxy \rightarrow Q^{*=}ax \wedge Q^{*=}yb$. I think this must somewhere have gotten discarded, perhaps because the admissibility of this assumption is not so obvious. Indeed, the technique needed to show that it is legitimate is the one I deploy below: Consider $Q\xi\zeta \wedge Q^{*=}\zeta b$. (And see also note 24.)

Note also that the argument just given is *not* intended to demonstrate the extensional correctness of the definition of $R^{=}_{*}$. That is, it is not intended as a proof that $R^{=}_{*}ab$ iff b is an R-descendant of a. The definition of $R^{=}_{*}$ is supposed to constitute a rigorous analysis of the ordinary notion of the ancestral. The *ex*tensional correctness of that definition is then meant to follow without proof from its *in*tensional correctness. What the argument just given is supposed to show is simply that $R^{=}_{*}$ and $R^{*=}$ are provably equivalent, which suffices to establish the *ex*tensional correctness of Frege's original definition of the ancestral, as well, without any appeal to arithmetical induction. So, if we wish, we can continue to proceed with Frege's definition, confident now in its extensional correctness, but on grounds that do not invite objections of circularity.

Of course, there is an obvious question to ask about the definition of $R^{=}_{*}$ and my claim that it is intensionally correct, namely: To what in the ordinary notion of an R-descendant does the condition of Dedekind finitude correspond? But there is an easy answer to that question: When we say that there is a path from a to b, what we mean is that we could actually get from a to b by following that path. If the path were Dedekind infinite, however, then taking the first step along that path would leave us with just as many steps to take as we had before we started. But then we *cannot* get from a to b by following this path: We can take some R-steps, and we can keep right on taking them, but we are never going to get to b, because we are not actually making any progress toward b. And if that still does not seem sufficiently intuitive, then note that this very idea turns up in the hymn "Amazing Grace", where eternity is described in these words:[27]

> When we've been here ten thousand years
>
> Bright shining as the sun,
>
> We've no less days to sing God's praise
>
> Than when we'd first begun.

[27] This particular verse was not written by the reformed slave trader John Newton, who was originally responsible for the hymn. Ironically, it originated in the spiritual tradition of the African-Americans he'd helped to enslave. It was being used as a verse of "Amazing Grace" at least as early as 1852, when it appears as such in Harriet Beacher Stowe's classic book *Uncle Tom's Cabin*. The words themselves date to at least 1790, however, when they appear in *A Collection of Sacred Hymns*, though as a verse of a different song. It was, and still is, common for verses to be recycled in this way in African-American spirituals. (Thanks to Wikipedia for some of this information.)

And, as Sam Wheeler mentioned to me, the same sort of idea is at work in the paradoxes of Zeno: The problem Achilles has is that, even once he is halfway to the goal, he has just as many tasks left to complete as he did before he started.

One might object, however, that this shows only that the condition of Dedekind finitude is *necessary*, not that it is sufficient.[28] The intuition that is being used to motivate the condition is that an R-path from a to b must, as we might put it, be *completable*. And it is clear enough that a completable path must be Dedekind finite. Is it so clear, however, that every Dedekind finite path is completable? Intuition is not going to answer that question, which means we need a proof that every Dedekind finite path is completable. One might well suspect that any such proof would have to use induction. And, if so, then that would re-instate the Papert–Parsons objection.

In fact, however, matters are not nearly so dire. The objection we are considering concedes that the definition of $R^=_*$ articulates necessary conditions on the existence of an R-path and so concedes that the conditional

(NEC) If b is an R-descendant of a, then $R^=_* ab$

can be established by reflection. As a glance back at page 6 will show, however, that was always the problematic direction: It was for the proof of

(NEC*) If b is an R-descendant of a, then $R^* ab$

that we required ordinary arithmetical induction.[29] We did not need *arithmetical* induction to prove the converse

(SUF*) If $R^* ab$, then b is an R-descendant of a

but only the sort of induction justified by Frege's definition of the ancestral.[30] But that now implies that

(SUF) If $R^=_* ab$, then b is an R-descendant of a

[28]Special thanks to Øystein Linnebo here.

[29]The kind of induction you have available is determined by what is in the antecedent, since that is what you will be supposing for purposes of the proof.

[30]The proof does, of course, depend upon some claims about R-descendants, namely: Any immediate R-descendant of a is an R-descendant of a; and any immediate R-descendant of an R-descendant of a is also an R-descendant of a. Those are plausibly conceptual truths, however.

can also be proven without any appeal to arithmetical induction: We need only put the proof of (SUF*) together with the proof just given that $R^{=}_{*}ab$ iff $R^{*=}ab$. So the Papert–Parsons objection is not, in fact, re-instated.

The argument just given is of a type first introduced by Georg Kreisel (1972): a so-called 'squeezing argument'.[31] The general structure of such arguments is as follows (Smith, 2011, §1). Suppose we have some informal notion $\mathfrak{I}$ and that we want to show that some rigorous notion $\mathfrak{R}$ provides an extensionally correct analysis of $\mathfrak{I}$. Suppose further that it is uncontroversial that $\mathfrak{R}$ provides a *necessary* condition for $\mathfrak{I}$. Then one way to show that $\mathfrak{R}$ is extensionally adequate is to find some other rigorous notion $\mathfrak{R}'$ that uncontroversially provides a *sufficient* condition for $\mathfrak{I}$ and then to show rigorously that $\mathfrak{R}$ is sufficient for $\mathfrak{R}'$, thus 'squeezing' $\mathfrak{I}$ between $\mathfrak{R}$ and $\mathfrak{R}'$. To put the point set-theoretically, we are supposing that it is uncontroversial that:

$$\mathfrak{R}' \subseteq \mathfrak{I} \subseteq \mathfrak{R}$$

and that rigorous argument assures us that

$$\mathfrak{R} \subseteq \mathfrak{R}'$$

from which it then follows that $\mathfrak{I} = \mathfrak{R}$. In our case, $\mathfrak{I}$ is the intuitive notion of an R-descendant; $\mathfrak{R}$ is the ancestral as Frege defines it; and $\mathfrak{R}'$ is the ancestral as I have defined it. So we are 'squeezing' the intuitive notion between Frege's definition of the ancestral and mine.

One might yet want to object, of course, that this does not show that the offered definition of $R^{=}_{*}$ is a 'correct analysis' of the ordinary notion of an R-descendant: The condition of Dedekind finitude has been shown to be mathematically sufficient but not to be intensionally sufficient. Maybe. I'm not sure. I'm inclined to think that, even if the analysis leaves out some condition that the intuitive notion includes (completability, say), if we can prove that this additional condition is redundant, the analysis still has a reasonable claim to intensional correctness. To be honest, though, I would be uncomfortable putting too much weight on the claim that the definition is intensionally correct: For the usual sorts of reasons, I am far from certain that there are any 'correct analyses' of ordinary notions. That does not imply, however, that there are never any interesting or important differences between extensionally equivalent definitions of a given notion. Nor does it imply that we cannot demand more of a

[31] Thanks here to an anonymous referee for pointing out this similarity.

definition of natural number, if it is to have the sort of epistemological interest Frege wanted his definition to have, than that it should be extensionally correct. And the present definition of $R^{=}_{*}$ has, in that sort of respect, a number of advantages over Frege's definition of $R^{*=}$.

For present purposes, the key feature of the definition of $R^{=}_{*}$ is that it allows us to prove the principle of arithmetical induction, *as that principle is ordinarily understood*, without any appeal to arithmetical induction. The proof is a version of the one considered earlier: I propose to derive

(IND) $$\forall F[F0 \wedge \forall x \forall y(\mathbb{N}x \wedge Fx \wedge Pxy \rightarrow Fy) \rightarrow \forall x(\mathbb{N}x \rightarrow Fx)]$$

from

(ANC′) $$\forall F[F0 \wedge \forall x \forall y(P^{=}_{*}0x \wedge Fx \wedge Pxy \rightarrow Fy) \rightarrow \forall x(P^{=}_{*}0x \rightarrow Fx)]$$

and

(EQ′) $$\forall x(P^{=}_{*}0x \equiv \mathbb{N}x)$$

Frege's definition of the ancestral and our proof that $R^{=}_{*}ab$ iff $R^{*=}ab$ together imply (ANC′). And (EQ′) follows from (NEC) and (SUF), given the observation—which is of course presupposed by everything we are doing here—that the natural numbers are the P-descendants of 0.

4 Simplifying the Definition of Natural Number

It is worth considering more closely how the defintion of an R-descendant applies to the case of natural numbers specifically.

If we take $\mathbb{N}n$ to be defined as $P^{=}_{*}0n$, then that amounts to:

$$\begin{aligned}\mathbb{N}n \equiv 0 = n \vee \exists Q[&\forall x \forall y(Qxy \rightarrow Pxy) \wedge \\ &\forall x \forall y \forall z(Qxy \wedge Qxz \rightarrow y = z) \wedge \\ &\forall x \forall y \forall z(Qxz \wedge Qyz \rightarrow x = y) \wedge \\ &\exists x(Q0x) \wedge \neg \exists x(Qx0) \wedge \\ &\exists x(Qxn) \wedge \neg \exists x(Qnx) \wedge \\ &(\exists y(Qxy) \wedge x \neq 0 \rightarrow \exists z(Qzx)) \wedge \\ &(\exists y(Qyx) \wedge x \neq n \rightarrow \exists z(Qxz)) \wedge \\ &\neg \mathbf{Eq}_x(\exists y(Qxy \vee Qyx); \exists y(Qxy \vee Qyx) \wedge x \neq 0)]\end{aligned}$$

Some of these conditions are now guaranteed to be satisfied, however, due to what we know about P. In particular, Q has to be one-one, since P is; and, since $\neg\exists x(Px0)$, we must have $\neg\exists x(Qx0)$, as well. So the definition reduces to:

$$\begin{aligned}\mathbb{N}n \equiv 0 = n \vee \exists Q[&\forall x\forall y(Qxy \to Pxy) \wedge \\ &\exists x(Q0x) \wedge \\ &\exists x(Qxn) \wedge \neg\exists x(Qnx) \wedge \\ &(\exists y(Qxy) \wedge x \neq 0 \to \exists z(Qzx)) \wedge \\ &(\exists y(Qyx) \wedge x \neq n \to \exists z(Qxz)) \wedge \\ &\neg\mathbf{Eq}_x(\exists y(Qxy \vee Qyx); \exists y(Qxy \vee Qyx) \wedge x \neq 0)]\end{aligned}$$

And if one thinks about the remaining conditions, then it is easy to see that they all concern the domain and range of the relation Q. In fact, they can be expressed in terms of conditions on its field, as follows:[32]

$$\begin{aligned}\mathbb{N}n \equiv 0 = n \vee \exists F[&F0 \wedge Fn \wedge \\ &\forall x\forall y(Fy \wedge Pxy \to Fx) \wedge \\ &\forall x\forall y(Fx \wedge Pxy \wedge x \neq n \to Fy) \wedge \\ &\neg\mathbf{Eq}_x(Fx; Fx \wedge x \neq 0)]\end{aligned}$$

The idea is that we can recover $Q\xi\zeta$ itself, if we wish, as: $P\xi\zeta \wedge F\xi \wedge F\zeta$. The first two conditions then correspond to $\exists x(Q0x)$ and $\exists x(Qxn)$; the third, to $\exists y(Qxy) \wedge x \neq 0 \to \exists z(Qzx)$; and the fourth, to $\exists y(Qyx) \wedge x \neq n \to \exists z(Qxz)$ and, to some extent, $\neg\exists x(Qnx)$, which one can now see is not essential. In fact, it is clear that the third condition $\forall x\forall y(Fy \wedge Pxy \to Fx)$ is not essential, either. Moreover, we need not treat the case $n = 0$ specially, since we can take $F\xi$ to be: $\xi = 0$, and the rest of the conditions will be satisfied.

So the definition of natural number simplifies to:

$$\begin{aligned}\mathbb{N}n \equiv \exists F[&F0 \wedge Fn \wedge \\ &\forall x\forall y(Fx \wedge Pxy \wedge x \neq n \to Fy) \wedge \\ &\neg\mathbf{Eq}_x(Fx; Fx \wedge x \neq 0)]\end{aligned}$$

That is: n is a natural number if there is a Dedekind finite concept that is true of both 0 and n and is closed under successors, except that it need

[32]This only works because P is one-one. One cannot reframe the original definition of

not be true of the successor of n. The point, obviously, is that, if n is not finite, then the two conditions $F0$ and $\forall x \forall y(Fx \wedge Pxy \wedge x \neq n \rightarrow Fy)$ will force F to be true of all natural numbers and so to be Dedekind infinite.

The proof of induction from this definition is relatively straightforward. First, we establish:

(6) $$\mathbb{N}n \rightarrow \forall F[F0 \wedge \forall x \forall y(Fx \wedge Pxy \rightarrow Fy) \rightarrow Fn]$$

Suppose that $F0$ and $\forall x \forall y(Fx \wedge Pxy \rightarrow Fy)$, but $\neg Fn$. We want to show that $\neg \mathbb{N}n$, for which we need to show that, for any G, if $G0$, Gn, and $\forall x \forall y(Gx \wedge Pxy \wedge x \neq n \rightarrow Gy)$, then G is Dedekind infinite. It is enough to show, under those hypotheses, that:

(7) $$P^{*=}0a \rightarrow Ga$$

where, note, that is Frege's version of the weak ancestral (which, again, we can treat as just an abbreviation). Frege himself shows that $P^{*=}0\xi$ is Dedekind infinite, so (7) will imply that $G\xi$ is Dedekind infinite, as well.[33]

To prove (7), we use (Gg 152) and so need to establish that $G0$, which we have assumed, and:

$$\forall x \forall y[P^{*=}0x \wedge Gx \wedge Pxy \rightarrow Gy]$$

We have assumed that

$$\forall x \forall y(Gx \wedge Pxy \wedge x \neq n \rightarrow Gy)$$

so it will be enough to establish that, if $P^{*=}0x$, then $x \neq n$. So suppose $P^{*=}0x$. Since we have also supposed that $F0$ and $\forall x \forall y(Fx \wedge Pxy \rightarrow Fy)$, we have by (Gg 144) that Fx. But then $x \neq n$, since we have also supposed that $\neg Fn$. So that establishes (7) and therefore also (6).[34]

Now, as said earlier, induction is really the stronger principle:

(8) $$\mathbb{N}n \rightarrow \forall F[F0 \wedge \forall x \forall y(\mathbb{N}x \wedge Fx \wedge Pxy \rightarrow Fy) \rightarrow Fn]$$

$R^{=}_{*}$ is similar terms.

[33] In Frege's system, the fact that F is Dedekind infinite can be concisely expressed as: $P^{*=}(\mathrm{N}x : Fx, \mathrm{N}x : Fx)$. So the fact that $P^{*=}0\xi$ is Dedekind infinite is Theorem 165, and the fact that every superset of a Dedekind infinite set is Dedekind infinite is Theorem 476.

[34] Special thanks to an anonymous referee for pointing out a thinko in an earlier version of this proof.

But, to prove this, we may proceed in much the way Frege does, namely, by taking $F\xi$ to be $\mathbb{N}\xi \wedge F\xi$ in (6). That gives us:

$$\mathbb{N}n \rightarrow [(\mathbb{N}0 \wedge F0) \wedge \forall x\forall y((\mathbb{N}x \wedge Fx) \wedge Pxy \rightarrow (\mathbb{N}y \wedge Fy)) \rightarrow Fn]$$

Again, $\mathbb{N}0$ is trivial, so what we need to show is that:

$$\forall x\forall y(\mathbb{N}x \wedge Fx \wedge Pxy \rightarrow Fy)$$

implies:

$$\forall x\forall y((\mathbb{N}x \wedge Fx) \wedge Pxy \rightarrow (\mathbb{N}y \wedge Fy))$$

And for that, it will suffice to prove that, if $\mathbb{N}x$ and Pxy, then $\mathbb{N}y$. But that is easy—I'll leave it as a simple exercise—so we are done.

5 Concluding Remarks

It is important to see that Frege's definition of $R^{*=}$ has some advantages of its own over the present definition of $R^{=}_{*}$. As mentioned above, Frege's definition of $R^{*=}$ is Π^1_1. The definition of $R^{=}_{*}$, by contrast, is Σ^1_2: $\exists F\forall R\phi$, the universal second-order quantifier coming from $\neg\mathrm{Eq}_x(\ldots;\ldots)$, since Eq is Σ^1_1: $\exists R\phi$. Frege's definition is, in this respect, best possible, since the ancestral cannot be defined by a Σ^1_1 formula[35] and so is not even Δ^1_1.

The additional complexity of the definition of $R^{=}_{*}$ surfaces in the logical resources required for proofs involving it. The transitivity of $R^{*=}$ can be proven without any appeal to comprehension (Boolos, 1998, p. 159). So far as I can tell, however, the proof that $R^{=}_{*}$ is transitive needs a fair bit of comprehension, the complication being that, if we have $R^{=}_{*}ab$ and $R^{=}_{*}bc$, then c may already be on the path from a to b. In that case, we cannot just paste these paths together, but have to truncate the first one, much in the way we did in the proof that $R^{*=}ab$ implies $R^{=}_{*}ab$. For similar reasons, the proof that $R^{*=}ab$ implies $R^{=}_{*}ab$ uses Π^1_2 comprehension (though the proof of the converse seems to use just Π^1_1 comprehension). The proof of induction from the simplified

[35]This follows from the compactness theorem for first-order logic. Suppose that $\exists F_1 \ldots \exists F_n\phi(a, b, R, F_1, \ldots, F_n)$ holds if, and only if, there is a finite sequence $a = a_0, a_1, \ldots, a_n = b$ where Ra_ia_{i+1}. Consider the first-order formula $\phi(a, x, R, F_1, \ldots, F_n)$ together with $\neg Rab$, $\neg\exists x_0(Rax_0 \wedge Rx_0b)$, $\neg\exists x_0\exists x_1(Rax_0 \wedge Rx_0x_1 \wedge Rx_1b)$, etc. Obviously, there are models of any finite subset of these and $\phi(a, x, R, F_1, \ldots, F_n)$. By compactness, there is thus a model in which all of them hold. In that model, there is no finite sequence connecting a to b, yet $\phi(a, x, R, F_1, \ldots, F_n)$ holds so $\exists F_1 \ldots \exists F_n\phi(a, x, , R, F_1, \ldots, F_n)$ holds as well. Contradiction.

definition of $\mathbb{N}\xi$ discussed in the last section (which is also Σ^1_2) also uses Π^1_2 comprehension, though only in deriving the stronger form of induction (8) from the weaker one (6)—in particular, when we use induction on $\mathbb{N}\xi \wedge F\xi$—not in the argument for (6) itself.

But these advantages are mostly technical, and my purpose here has not primarily been technical. My goal, rather, has been to show that there is a way of defining the concept of natural number in second-order logic that is immune to the sort of worry first voiced by Poincaré: that any definition of an enumerative notion of finitude must in some way make use of that very notion. The strongest way to put the point of these investigations would be that the definition of $R^=_*$ captures the connection between the relations *parent* and *ancestor*, or *successor* and *natural number*, in a way that is *intensionally* correct. But even if that claim is too strong, we have seen that the definition of $R^=_*$ nonetheless makes it possible to prove its equivalence with the ordinary notion of an R-descendant without appeal to arithmetical induction, and so makes it possible to prove arithmetical induction, as it is ordinarily understood, without any need to appeal to arithmetical induction. And that allows us to answer the best of the circularity objections: the one that Parsons attributes to Papert.

I thus take myself to have shown that arithmetical induction can be justified in much the way Frege supposed, though we need to use a definition different from his to do it.[36]

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