Truth in Frege

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Tarski is widely regarded as the father of formal semantics, and rightly so. But if Tarski is the father, then Frege is the grandfather. Frege's *Grundgesetze der Arithmetik* contains a semantic theory for his formal language, his begriffsschrift (or 'conceptual notation'), that is no less rigorous than the one for the calculus of classes that Tarski develops in the early parts of "The Concept of Truth in Formalized Languages" (Tarski, 1958). Like Frege's semantics for begriffsschrift, Tarski's is stated in an informal meta-theory and is in no sense 'formal'. Moreover, Frege argues in section 31 of *Grundgesetze* that his semantics is adequate to assign a unique denotation to every expression of begriffsschrift, and this argument has essentially the same purpose, and much the same structure, as Tarski's proof that his semantics is materially adequate.[1]

Of course, there are differences between Frege and Tarski. Most of these derive from the fact that they have very different reasons for being interested in semantic theory. Frege wouldn't have seen much point in formalizing the semantics for his formal langauge,[2] since its purpose was not to report the pre-established meanings of the expressions of that language but rather to establish those meanings. Formalizing the semantic theory for begriffsschrift in begriffsschrift would therefore have been unhelpful. Tarski, on the other hand, is quite plainly interested in formalized semantic theories, and, if he does not present the semantics for the calculus of classes as a formal theory, then that is because it would have been obvious enough to his readers how it could be formalized.

[1] Unfortunately, space constraints require that much of what we say here be dogmatically asserted. For defense of this particular dogmatic assertion, see Heck's book *Reading Frege's* Grundgesetze (Heck, 2012, Part I) and the papers on which that material was based (Heck, 1998, 1999, 2010).

[2] Some commentators would argue that Frege would have found the very project of formal semantics unintelligible. This is often taken to be at the crux of his dispute with Hilbert. We disagree (Antonelli and May, 2000), and our claim here is much weaker.

Indeed, Tarski would have expected his readers already to be familiar with the general structure of semantic theories, much as, in his paper on the completeness theorem, Gödel (1930) exepected his readers to be familiar with the structure of models of first-order theories. Both expected their readers to be familiar with the technical background that made the discussion intelligible.

The later parts of "The Concept of Truth" are, of course, focused on formalized semantic theories, primarily on ones formalized in the theory of types. But Tarski's central interest is not so much in the structure of semantic theories as it is in the question whether such theories can be immunized from the semantic paradoxes. What made that issue pressing was the fact that, by the time Tarski did his work, the notion of truth had assumed a central place in logical theory; it is in terms of truth, for example, that logical validity is defined. Yet the notion of truth was subject to paradoxes that could not be ignored forever, especially after Gödel (1931) showed us, in his paper on the *in*completeness theorems, that the self-reference needed for the paradoxes was available already in elementary arithmetic. For Tarski, then, formalizing semantic theories—and clarifying the semantics of quantification, via the notion of satisfation—was just one step towards his real goal: showing how semantic theories can be consistently developed.

Frege, by contrast, never mentions the semantic paradoxes and was not interested in 'theories of truth' in the way Tarski, Kripke (1975), and so many others since have been (this volume, *passim*). That is very much not to say that Frege was not interested in truth. Frege was profoundly interested in logic, and, on his view, truth is the real subject-matter of logic. As he puts it in a famous passage:

> Just as "beautiful" points the way for aesthetics and "good" for ethics, so does the word "true" for logic. All sciences have truth as their goal; but logic is also concerned with it in a quite different way: logic has much the same relation to truth as physics has to weight or heat. To discover truths is the task of all sciences; it falls to logic to discern the laws of truth. (Tht, op. 58)[3]

Given this conception of logic, it should be no surprise that, among the many tasks Frege sets himself in Part I of *Grundgesetze*, in which he

[3] References to Frege's published papers are given by the page number in the original publication. Most reprintings include this information.

explains his formal system, is that of proving that each of his 'Basic Laws' is true and that each of his rules of inference preserves truth (Heck, 2010; 2012, Ch. 2). The semantic theory Frege gives for begriffsschrift plays a critical role in these arguments. It is because of the particular semantic clause that governs the conditional, for example, that Frege's Basic Law I[4]

$$\vdash p \to (q \to p)$$

is true and that his first method of inference, *modus ponens*, is valid. Frege's arguments for the truth of Law I (Gg, v. I, §18) and for the validity of *modus ponens* (Gg, v. I, §14) appeal directly, and only, to that clause.

For Frege, then, truth is the most fundamental notion for both logic and semantics. This doctrine surfaces most directly in Frege's distinctive, and oft misunderstood, claim that the reference of a sentence is its truth-value. As we shall see, this claim has two complementary aspects: It underlies Frege's truth-functional understanding of the sentential operators, and it surfaces in his treatment of concepts as functions from objects to truth-values.[5] We shall also discuss Frege's seemingly puzzling claim that the truth-values are objects, in an effort to make it somewhat less puzzling. We shall then turn our attention to the brief remarks Frege does make about the 'truth-predicate'. Finally, we shall consider, very briefly, the question whether Frege was a deflationist.

1 Sentences Refer to Truth-values

As just said, Frege's most important thesis about truth is that the reference of a sentence is its truth-value. We shall call this the 'Truth-Value Thesis'.

Frege himself does not clearly distinguish the Truth-Value Thesis from the thesis that truth-values are *objects*. As Dummett has emphasized,[6] since, for Frege, ontological categories supervene on syntatic ones, this latter thesis is all but equivalent to the claim that sentences are a sort of proper name. It should be obvious that this syntactic thesis is independent of the Truth-Value Thesis. We will discuss the claim that truth-values are objects in Section 4.

[4] We shall silently translate Frege's notation into modern notation, only preserving his prefixed vertical "judgement" and horizontal "content" strokes.

[5] Unfortunately, we shall not have space to discuss Frege's truth-conditional conception of meaning, which is another aspect of the thesis.

[6] The various claims of Dummett's we shall mention come primarily from *Frege: Philosophy of Language* (Dummett, 1981), especially chapter 12.

As Dummett also makes clear, the claim that sentences *refer*, while it sounds odd to the modern ear, should not be doubted on that ground.[7] Frege's notion of reference, though grounded intuitively in the relation between a name and its bearer, is a technical one, to be explicated in terms of the theoretical work it is supposed to do. That work is semantic. So the Truth-Value Thesis amounts to the claim that the fundamental semantic fact about a sentence is that it is true or false: that it has whatever truth-value it has. Dummett therefore suggests that we state Frege's view as: the 'semantic value' of a sentence is its truth-value. This usage has become almost standard in truth-theoretic semantics (see, e.g., Larson and Segal, 1995). But we shall defer to Frege's usage and so will continue to speak of sentences as 'referring' to truth-values.

It is a common complaint that Frege gives little direct argument for the Truth-Value Thesis. The arguments he does give are in "On Sense and Reference", and they are clearly inadequate. The first argument is that it only matters to us whether a name refers in so far as we are concerned with the truth-value of some sentence in which it occurs. That may be true, but it hardly "drive[s us] into accepting the truth-value of a sentence as constituting what it refers to" (SM, op. 34).

A second set of considerations follows:

> If our supposition that the reference of a sentence is its truth-value is correct, then the latter must remain unchanged when a part of the sentence is replaced by an expression with the same reference. And this is in fact the case. . . . If we are dealing with sentences for which the meaning of their component parts is at all relevant, then what feature except the truth-value can be found that belongs to such sentences quite generally and remains unchanged by substitutions of the kind just mentioned? (SM, op. 35)

This looks like an attempt to derive the Truth-Value Thesis from the compositionality of reference, that is, from the claim that the reference of a complex expression is determined by the references of its parts. But it too is unconvincing. What it shows is that it is *consistent* with compositionality to take sentences to refer to their truth-values. If Frege cannot think of another option, then that, one might think, is his

[7] Frege's German term is *bedeuten*, whose most natural translation is "meaning". But that doesn't help, since the claim that sentences 'mean' their truth-values sounds odd for a different reason.

problem.[8]

Why are Frege's arguments for the Truth-Value Thesis so pathetic? The answer is simple: He doesn't really have a direct argument for it. His argument is ultimately pragmatic, as he indicates in *Grundgesetze*: "How much simpler and sharper everything becomes with the introduction of truth-values, only detailed acquaintance with this book can show" (Gg, p. x). The Truth-Value Thesis solves a lot of problems, and it solves them, Frege thinks, better than anything else on offer. That is his real argument.

Given Frege's commitment to the compositionality of reference, the Truth-Value Thesis has two complementary aspects, since sentences both *have parts* themselves and *are parts* of other sentences. Now, in Frege's logic, the sentential operators are negation and the conditional. So compositionality and the Truth-Value Thesis imply that negation and the conditional are truth-functional. That is the first of the two complementary aspects. The other is the thesis that the reference (semantic value) of a predicate is a function from objects to truth-values. As Dummett again makes clear, this all but follows from the Truth-Value Thesis, given three of Frege's other commitments: that the reference of a predicate is a function; that names refer to objects; and that the reference of a sentence "Fa" is the result of applying the function to which "$F\xi$" refers to the object to which "a" refers.[9] For consider such a sentence. We know that "$F\xi$" must denote a function and that "a" must denote an object. So the arguments to the function "$F\xi$" are certainly objects, and if the reference of "Fa" is to be a truth-value, then the values of the function must be truth-values. So "$F\xi$"' denotes a function from objects to truth-values. Such functions are what Frege calls 'first-level concepts'.[10]

In the presence of compositionality, the Truth-Value Thesis thus entails that sentential operators are truth-functional and that concepts are functions from objects to truth-values. And now the important

[8] And, of course, there is another option: The Russellian proposition expressed by the sentence will remain unchanged.

[9] This last doctrine reflects Frege's view that semantic composition is 'internal' to the semantics of predicates, in a sense we try to explain elsewhere (Heck and May, 2006, 2013).

[10] Frege holds similar views about predicates of other logical types. For example, a two-place predicate like "ξ loves η" refers, on Frege's view, to a two-place function from objects to truth-values: to a two-place, first-level concept. Higher-level concepts take concepts (more generally, functions) as arguments. Quite generally, then, predicates refer to functions from values of some appropriate type (or types) to truth-values.

observation is that *neither* of these claims is present in Frege's early work. Rather, they emerge as responses to problems Frege had, problems that result from the failure of an earlier view. So, to understand why Frege adopts the Truth-Value Thesis, we need to understand those problems and how the Truth-Value Thesis solves them.

2 Concepts Are Functions From Objects to Truth-Values

Let us focus first on Frege's doctrine that concepts are functions from objects to truth-values. The idea that concepts are functions is already present in *Begriffsschift*. The distinction between function and argument is there presented as the key to Frege's new analysis of generality, and Frege explains his distinction between function and argument in *Begriffsschrift* (Bg, §9) almost exactly as he explains the distinction between concept and object in his mature work (FC, op. 6). Frege does not use the term "concept" in *Begriffsschrift*, but when he introduces the distinction between function and argument, he does so by applying it to the sentence "Hydrogen is lighter than carbon dioxide". Frege says that, if we regard "hydrogen" as replaceable by other expressions, then "…'hydrogen' is the argument and 'being lighter than carbon dioxide' is the function…" (Bg, §9). Obviously, "being lighter than carbon dioxide" is a predicate, and so the upshot is that we are to regard it, logically, as a function.

Frege thus regarded a simple sentence like "Bob runs" as being analyzable into a function, "runs", and an argument, "Bob". For us, the next question to ask would be: What are the inputs and outputs of this function? In *Begriffsschrift*, however, the question does not arise, since Frege's official view there is that functions are *expressions* (Bg, §9). Frege abandons this view, however, by 1881 (Heck and May, 2010, 2013), and then the question does arise what the arguments and values of these 'concept-functions' are. It is clear enough what the arguments must be: Frege held already in *Begriffsschrift* that the content of a proper name is the object it denotes (Bg, §8), so the arguments are just ordinary objects. But what are the *values* of concept-functions supposed to be?

In his writings from the early 1880s, Frege does not explicitly answer this question, but it is easy enough to deduce the answer. Consider a simple sentence "Fa". Applying the concept-function that is the content of "$F\xi$" to the object that is the content of "a" should yield the content of the complex expression that they together constitute. But Frege's view in the early 1880s was that the content of "Fa" is a content "that can become a judgement" (Bg, §2), for short, a judgeable content. So Frege's original view must have been the following: A proper name has as its content an object; a predicate has as its content a function from objects to judgeable contents; and the content of "Fa", which is a content that can become a judgement, is the result of applying the function that is the content of "F" to the object that is the content of "a".[11]

This view is really quite elegant, not to mention strikingly reminiscent of Russell's later notion of a propositional function (Russell, 1903, Ch. VII). Unfortunately, as a now famous argument shows, it has consequences that would have been unacceptable to Frege. If the content of

(1) The Evening Star is a planet

is the result of applying the function that is the content of "ξ is a planet" to the object that is the content of "the Evening Star", then

(2) The Morning Star is a planet

must have the same content. But (1) and (2) cannot have the same content, not if identity of content is to guarantee identity of logical properties, as Frege clearly insists it must:[12]

> ...[T]he contents of two judgements may differ in two ways: either the consequences derivable from the first, when it is combined with certain other judgements, always follow from the second, when it is combined with these same judgements, or this is not the case. ...I call that part of the content that is the *same* in both the *conceptual content*. ...Everything necessary for a correct inference is expressed in full, but what is not necessary is generally not indicated; *nothing is left to guesswork*. (Bg, §3, emphasis in original)

And now, since (1) and (2) have different logical consequences, they must have different contents.

How can Frege avoid the conclusion that (1) and (2) must have the same content? The conclusion depends upon just four assumptions:

[11] Beaney (2007) comes to the same conclusion.

[12] Some readers—e.g., Brandom (1994, pp. 94ff) and Kremer (2010)—have claimed to find a stronger thesis here: that, if A and B have the same consequences, then they must also have the same content. We don't read the passage that way, but the issue is not relevant at present.

i. The content of a proper name is its bearer.

ii. Concepts are functions.

iii. The content of a simple sentence "Fa" is the result of applying the concept-function that is the content of "F" to the content of "a".

iv. Logical properties are determined by content, so that sameness of content implies sameness of logical properties.

Only (iii) is dispensible: (ii) is the key to Frege's understanding of logical generality; (iv) is integral to Frege's conception of logic and its relation to content (May, 2006); (i) is central to Frege's understanding of identity as objectual. This last view is not present in *Begriffsschrift* but is in place by 1881 and is fundamental to Frege's logicism (May, 2001; Heck and May, 2006; Heck, 201X). So, as said, (iii) is what Frege must abandon: The content of "Fa" *cannot* be the result of applying the concept-function that is the content of "F" to the object that is the content of "a". To put it differently: The values of concept-functions *cannot* be judgeable contents. So what are they?

Frege's mature view, of course, is that the values of concept-functions are truth-values: Concepts are functions from objects to truth-values. This thesis is thus partly a result of Frege's being forced to re-think the question what the values of concept-functions are, once he realizes that they cannot be judgeable contents.

One source of Frege's new view is how he proposes to distinguish concepts from objects. In a letter to the philosopher Anton Marty, written in 1882, Frege remarks that he "regard[s] it as essential for a concept that the question whether something falls under it has a sense", a question that would be senseless in the case of an object (PMC, p. 101). The view that concepts are functions from objects to truth-values fits naturally with this suggestion, since what such a function does is sort objects into two baskets: those that fall under the concept and those that do not.

Such a consideration does make the view that the truth-values are the values of concept-functions attractive, but more would have been needed to drive Frege to it, because it has deeply counter-intuitive consequences.[13] In particular, it implies that concepts are extensional, so that there can be only one concept true of a given collection of objects. For example, if, as philosophers' lore has it, the animals that are supposed

[13] The argument to be given depends upon the assumption that functions themselves are extensional. We discuss this matter elsewhere (Heck and May, 2010, §2).

to have kidneys are the same as the animals that are supposed to have hearts, then the concepts *renate* and *cordate* will be the same concept, and that has no intuitive plausibility. Of course, the mature Frege does think that concepts are extensional. But that was not always his view, either.

The view that concepts are intensional is an almost immediate consequence of Frege's early view that concepts are functions from objects to judgeable contents. On that view, *is a renate* maps Bob to *Bob is a renate*, and *is a cordate* maps Bob to *Bob is a cordate*. Since the content *Bob is a cordate* has different logical properties from the content *Bob is a renate*, these are different contents, and so the functions *is a cordate* and *is a renate* are intensional: They have different values for the argument Bob and, indeed, for every argument. Moreover, in 1881, Frege is prepared to use the thesis that concepts are intensional as a premise—without saying even a word in defense of it—in one of his most important arguments, an argument for the conclusion that concepts must be distinguished from objects (BLC, p. 18). And yet, by 1884, Frege has changed his mind and come to regard concepts as extensional (Gl, §68, fn. 1). It is hard to imagine any explanation for this shift other than that Frege has, in the intervening years, changed his mind about what the values of concept-functions are: not judgeable contents, but truth-values.

So why that change? To answer this question, we need to look at the other aspect of the Truth-Value Thesis, the one connected with the fact that sentences can occur as parts of other sentences.

3 Sentential Connectives as Truth-Functions

Frege has sometimes been credited with the discovery of truth-tables (Kneale and Kneale, 1962, pp. 420, 531; Wittgenstein, 1979b, pp. 135ff), and something akin to truth-tables is indeed present in Frege's early work. Frege emphasizes that, when we are considering a binary sentential connective, we must distinguish four possible cases. But Frege does not mention the notion of truth in this connection: Frege does not say, in *Begriffsschrift*, that a conditional is false only when its antecdedent is true and its consequent is false. Frege's explanation of the conditional reads, rather, as follows:

> If A and B stand for contents that can become judgements. . . , there are the following four possibilities:

(1) A is affirmed and B is affirmed;
(2) A is affirmed and B is denied;
(3) A is denied and B is affirmed;
(4) A is denied and B is denied.

> Now $\vdash B \to A$ stands for the judgement that *the third of these possibilities does not take place, but one of the other three does*. (Bg, §5, emphasis in original; see also BLC, p. 35)

Frege does not speak of truth in his explanation of negation, either (Bg, §7). The suggestion that negation and the conditional express truth-functions is therefore wholly absent from *Begriffsschrift* and, indeed, from all of Frege's early writings.

Truth-functions appear in Frege's work for the first time in *Function and Concept*, published in 1891. After explaining his conception of functions as 'incomplete', Frege motivates his view that concepts are functions whose values are always truth-values (FC, opp. 13ff). But if the truth-values are admitted as values of functions, we can allow them to occur as arguments, too. Interesting cases then include negation and the conditional (FC, opp. 20ff), and Frege clearly and explicitly explains these as truth-functions. That said, so far as we know, nowhere in his later writings does Frege give the sort of 'tabular' account that Wittgenstein and the Kneales mention, so there is no real basis for attributing the discovery of truth-*tables* to Frege.[14]

Perhaps surprisingly, even the idea that the sentential operators express functions seems to be absent from *Begriffsschrift*. Its emergence is due to Frege's belated encounter with the work of George Boole, whom Frege seems to have read only after the appearance of scathing reviews of *Begriffsschrift* published by leading members of the Boolean school. Both Ernst Schröder (1972) and John Venn (1972) accuse Frege of essentially replicating Boole's work, but in a less satisfactory form and only partially.

Frege criticizes Boolean logic on several grounds. Perhaps his most penetrating criticism concerns Boole's distinction between the 'calculus of judgements' and the 'calculus of concepts'. The former is essentially what we know as propositional logic; the latter treats of relations between concepts, those expressed by "All F are G", "Some F are G", and so forth.

[14] Moreover, Frege never considers truth-tables for arbitrary formulae (e.g., $p \vee r \to q \vee r$), but only for the simplest cases, and there is no indication that he realized, as both Wittgenstein and Post (1921) did, that truth-tables can be used to determine the validity of an arbitrary propositional formula. As is now widely recognized, then, it is Wittgenstein and Post who deserve the real credit for the discovery of truth-tables.

Boole regards the calculus of concepts as basic and reduces the calculus of judgements to it. Frege argues against Boole on both points: He claims that Boole's attempt to reduce the calculus of judgements to the calculus of concepts is a failure and that Boole is wrong to treat the latter as basic.[15]

Boole's reduction proceeds as follows. Both calculi contain expressions of the forms "$A \times B$", "$A + B$", "$\bar{A}$", and so forth.[16] In the calculus of concepts, the letters are taken to denote classes (or extensions of concepts), and the operations are interpreted set-theoretically: Multiplication is intersection; addition is union; the bar represents complementation relative to the chosen 'universe of discourse'. How are the letters interpreted in the calculus of judgements? One might expect that they would denote truth-values, but that would be wrong:[17] They again denote *classes*, so the operations are still set-theoretic. In *The Laws of Thought*, for example, Boole takes the letters in the calculus of judgements to denote classes of times: the times a proposition is true. So a conditional (judgement) becomes a universal affirmative proposition (relation between classes): All times at which the antecedent is true are times the consequent is true (Boole, 1854, Ch. XI, §5).

Boole's central idea, of course, is to treat the sentential operators as expressing set-theoretic operations on the power set of some universe. What the universe comprises in the case of judgements varies, both in Boole's own work and in that of his followers, but that has proved not to be the crucial point. The crucial point is that, once we treat the sentential operators set-theoretically, the algebra so determined is (what we now call) a Boolean algebra, and it validates the laws of classical logic. Sadly, Frege could no more see the importance of this idea than Schröder could see the importance of Frege's new analysis of generality. But Frege is surely right that the attempted reduction of sentential logic to quantification theory is a failure, and not only for the case of "eternal truths such as those of mathematics" (BLC, p. 15). What is fundamental is sentential logic, and Frege goes so far

[15] Frege also points out that Boole has no way to mix the two, as in: $\forall x(Fx \to Gx) \vee \exists x(Gx \wedge Hx)$. And, of course, that Boole cannot handle nested quantifiers, as in: $\exists x \forall y(Rxy)$.

[16] The notation varies from logician to logician, as do the details of its interpretation, but these differences do not matter for our purposes.

[17] MacColl (1877, pp. 9–10) comes closest to this conception, but his official view is that the sentence-letters denote 'statements' . Schröder mentions MacColl in his review, but it is unclear if Frege ever read him. Frege mentions MacColl twice (AimCN, p. 93; BLF, p. 15), but what he says is all but lifted from Schröder.

as to describe himself, somewhat misleadingly, as reducing universal affirmative propositions to conditionals (BLC, pp. 17–18).

There is much more to be said about this, but just two points are important here. The first is that Boole does *not* treat the sentential connectives as expressing *truth*-functions. Boole does regard the case in which the universe contains just one element as important, but this is mostly because it makes the calculations in which he is interested especially simple. And the two elements of the algebra so determined, though denoted "1" and "0", are not the True and the False. As always in Boole, "1" denotes the universe, and "0" denotes the empty set, and the operations are set-theoretic.

The second point is that Boole undoubtedly *did* treat the sentential connectives as expressing *functions*. His use of the arithmetical expressions "$+$" and "$\times$" serves to emphasize this point. And, as critical as Frege is of Boole's over-reliance on the similarities between logic and arithmetic (AimCN, pp. 93–4; BLC, pp. 13–15), it is hard to imagine that Frege would not have been struck by this element of Boole's work. It might even have seemed like confirmation of his own emphasis on the importance of the notion of *function* to logic. Frege does not highlight this aspect of Boole's work in his critical pieces—he is too busy defending himself—but nor does he criticize it. So, although it may have taken him a little while to assimilate it, it seems very plausible that Frege got the idea that the sentential connectives express functions from his reading of Boole.

So we ask: What are the arguments and values of these functions? At one time, of course, the obvious thing for Frege to say would have been that the conditional expresses a two-place function from contents to contents. But, once Frege discovers the substitution argument, this option is off the table, since sentences cannot have judgeable contents as their semantic values.

So what are the arguments and values of the conditional, if they are not judgeable contents? The answer is almost there in *Begriffsschrift*. As we saw earlier, Frege distinguishes four possibilities:

(1) A is affirmed and B is affirmed
(2) A is affirmed and B is denied
(3) A is denied and B is affirmed
(4) A is denied and B is denied

and says that "$\vdash B \rightarrow A$ stands for the judgement that the third of these possibilities does not take place" (Bg, §5). This, of course, is wrong.

The language of affirmation and denial is not only quaint but misplaced, as Frege himself would eventually come to realize. This is essentially what Peter Geach (1965, p. 449) famously called "the Frege point". It is closely connected with what Frege himself called "the dissociation of assertoric force from the predicate" and regarded as one of his most important discoveries (WMR): If one asserts a conditional, then it is only the conditional as a whole that is affirmed; one need neither deny its antecedent nor affirm its consequent.[18] These contents are judge*able*, not judged. *Contra* Geach, however, the Frege point is not really there in *Begriffsschrift*. There are intimations of it, to be sure (Bg, §2). But if Frege had fully appreciated it, he could not have used the language of affirmation and denial the way he does in explaining the conditional.

The language of affirmation and denial is present in Frege's writings through 1882; it does not appear later. We speculate, therefore, that at some point between 1882 and 1884, Frege must have become dissatisfied with his use of such language, presumably for the reason we have just recalled. When he does, he needs a new way to explain the conditional; he needs to reconstruct his table of possibilities in terms of something other than affirmation and denial. In the writings from 1881 and 1882, Frege sometimes presents the table this way:

(1) A and B
(2) A and not-B
(3) not-A and B
(4) not-A and not-B

remarking that his "$B \to A$ denies the third" (BLC, p. 35). The language of affirmation and denial is thus almost gone. But this form of explanation will still be inadequate once Frege has decided that the conditional ought to be understood as expressing a function: We need to know what the arguments and values of that function are.

With this question in the front of his mind, and Boole's work in the back of his mind, Frege was eventually struck by an answer: What matters is not whether A and B are affirmed or denied but just whether they are true or false. And now the other piece falls into place: If the arguments and values of the connectives are truth-values, then the values of concept-functions must be truth-values, as well, since the values of concept-functions are also arguments of the connectives.

[18] Similarly, Frege's explanation of the conditional is insufficiently general, since it tells us only about *asserted* conditionals and does not seem to apply to embedded conditionals.

No doubt, there are other options. But that is why we said, earlier, that Frege's argument for the Truth-Value Thesis is ultimately pragmatic. Frege had a whole set of problems that were generated by the failure of his original view that sentences refer to judgeable contents. The Truth-Value Thesis not only solves those problems, it does so in an elegant way, not just tying the loose ends but tying them neatly. So now we can understand what Frege meant when he spoke of "[h]ow much simpler and sharper everything becomes with the introduction of truth-values" (Gg, p. x). Though he might have added: It would help to look at my earlier work to get a sense for what a mess things were before.

Of course, Frege's new view gives rise to problems of its own. We have already mentioned one counter-intuitive consequence: the extensionality of concepts. But there is an even worse problem:[19]

> ... [W]hat "$2^2 = 4$" means is the True just as, say, "2^2" means 4. And "$2^2 = 1$" means the False. Accordingly, "$2^2 = 4$", "$2 > 1$", and "$2^4 = 4^2$" all mean the same thing, viz., the True.... The objection here suggests itself that "$2^2 = 4$" and "$2 > 1$" nevertheless tell us quite different things, express quite different thoughts. (FC, op. 13)
>
> If now the truth-value of a sentence is its meaning, then on the one hand all true sentences have the same meaning and so, on the other hand, do all false sentences. From this we see that in the meaning of the sentence all that is specific is obliterated. (SM, op. 35)

4 Truth-values as Objects

We have argued that Frege's mature view that sentences refer to truth-values emerged between 1881 and 1884, when Frege was engaged with the work of the Booleans. It is important to understand that we are *not* claiming that Frege held already in 1884 that truth-values are *objects*.[20] As already noted, the two views are distinct. The ontological thesis that truth-values are objects is bound up with the syntactic thesis that

[19] We are here translating Frege's German term "*bedeuten*" (and its cognates) using the ordinary English equivalent, "mean", as one simply cannot appreciate the force of the objections Frege is considering otherwise.

[20] See we could agree with Goren Sundholm (2001, p. 62) that "Frege did not have the doctrine of (objectual) truth-values" until about 1890. Whether that is the right thing to say, however, is not so clear (Heck, 201X, §2).

sentences are proper names, and there is reason to believe that Frege had not arrived at this latter view by 1884. In particular, there are significant differences between his statements of the 'context principle' in *Die Grundlagen* (Gl, p. x and §62) and in *Grundgesetze* (Gg, v. I, §29). As Dummett (1981, ch. 12) has argued, these differences reflect a change from a view on which sentences have a privileged position to one on which they are but a kind of proper name (their syntactic complexity not withstanding). It is anyway reasonable to suppose that it took some time for Frege fully to adjust his conception of logic to his discovery of the truth-values. The puzzle mentioned at the end of the last section, for example, was one to which Frege did not have a satisfactory response until "On Sense and Reference", published in 1891 (Heck and May, 2010).

One might think, however, that Frege already treats sentences as proper names in *Begriffsschrift*. Just as in *Grundgesetze*, sentences occur in *Begriffsschrift* in positions where proper names occur. Frege is as happy to write

$$\vdash p \equiv \neg\neg p$$

as he is to write

$$\vdash A \equiv B,$$

where A and B names of geometrical points, as in the famous example in *Begriffssschrift* §8. Similarly, the variables "c" and "d" that occur in Frege's proposition (57)

$$\vdash c \equiv d \to (Fd \to Fc)$$

may be replaced either by singular terms or by sentences. For example, the proof of proposition (68)

$$\vdash \forall a(f(a)) \equiv b \to (b \to f(c))$$

requires the substitution of "$\forall a(Fa)$" for "c" in (57).

There is nonetheless a different sort of distinction between names and sentences in *Begriffsschrift*. Frege insists that "[w]hatever follows the content stroke must have a content that can become a judgement" (Bg, §2). So one cannot write "— A" unless A is a formula. Thus, in (68), b must be a formula, since it appears as antecedent of a conditional and so is preceded by the content stroke (as is apparent when the formula is given in Frege's notation). And, though Frege does not explicitly exclude such constructions, there is no evidence in *Begriffsschrift* that he would have regarded something with a sentence on one side of '$\equiv$' and a name on the other, such as

$$\forall a(Fa) \equiv 1,$$

as well-formed. Matters are very different in *Grundgesetze*, where

$$\forall a(a = a) = \dot{\epsilon}(-\epsilon),$$

which says that the True is the extension of the concept *is identical with the True*, is not just well-formed but true.

Why then does Frege treat the truth-values as objects?[21] It is often suggested that he has no choice but to do so, since, on his view, there are only two kinds of 'things', the complete and the incomplete. Objects are the paradigmatic complete entities; concepts and functions, the incomplete ones. Since the truth-values are not incomplete, they must be complete, and so must be objects. But this line of reasoning is at best inconclusive. There is no obvious reason Frege could not have held that there was more than one kind of complete entity.[22] Indeed, since, for Frege, ontological categories largely supervene on syntactic ones, the apparent grammatical differences between sentences and names speak strongly in favor of such a distinction.

If there were more than one type of complete entity, then the logic would become, in present-day terminology, 'many-sorted'. There would then be four types of one-place, first-level functions: from objects to objects; from objects to truth-values; from truth-values to objects; and from truth-values to truth-values. Similarly, there would be two different notions of identity. But we already have such distinctions in the hierarchy of types: Frege insists, for example, that we cannot really say that two functions are 'identical', since identity is a relation between objects (CSM, p. 121).

To understand why Frege treats truth-values as objects, and what his so treating them amounts to, we need to look closely at a particular feature of the formal development in *Grundgesetze*.

[21] It might be thought that there was a prior question, namely, why Frege treats truth-values as any kind of 'thing'. But this has already been answered: If concepts are to be functions from objects to truth-values, and if sentential connectives are to be truth-functional, then truth-values have to be able to occur as arguments and values of functions. So they have to be some kind of 'thing', and that view has to be in place by 1884, since it is what drives the view that concepts are extensional.

[22] Similar issues arise in connection with Frege's claim that numbers are objects (Heck, 2011).

Value-ranges are to functions as extensions are to concepts; think of a function's value-range as its graph (in the set-theoretic sense). Frege's Basic Law V governs the notation for value-ranges:

$$\vdash (\dot{\epsilon}(f\epsilon) = \dot{\epsilon}(g\epsilon)) \; = \; \forall x(fx = gx)$$

This says that the value-range of the function $f\xi$ is the same as that of $g\xi$ just in case they always have the same value for the same argument. One might have thought that this would only allow Frege to speak of the value-ranges of one-place functions and that he would need similar axioms for two-place functions and the like. But one of the most elegant features of the formal development in *Grundgesetze* is how Frege handles the value-ranges of two-place functions, namely, as what he calls 'double value-ranges'. Consider, for example, the function '$\xi + \eta$'. Fix its second argument and consider (e.g.) the function '$\xi + 2$'. The value-range of this function, $\dot{\epsilon}(\epsilon + 2)$, is the graph of the function whose value, for a given argument x, is $x + 2$. Suppose we now allow the second argument in $\dot{\epsilon}(\epsilon + 2)$ to vary; the resulting function, $\dot{\epsilon}(\epsilon + \eta)$, maps a given argument y to the value-range $\dot{\epsilon}(\epsilon + y)$. So what is the value-range of the function $\dot{\epsilon}(\epsilon + \eta)$? It is the *double* value-range $\dot{\alpha}\dot{\epsilon}(\epsilon + \alpha)$, the graph of the function whose value, for argument y, is the value-range $\dot{\epsilon}(\epsilon + y)$.[23] The analogue of Law V for two-place functions then emerges as a theorem:[24]

$$\vdash [\dot{\alpha}\dot{\epsilon}(f\epsilon\alpha) = \dot{\alpha}\dot{\epsilon}(g\epsilon\alpha)] \; = \; \forall x \forall y(fxy = gxy)$$

And, of course, the same construction allows Frege to use the double value-range $\dot{\alpha}\dot{\epsilon}(R\epsilon\alpha)$ of a relation as its extension: Extensions, quite generally, are just the value-ranges of concepts, that is, of functions from objects to truth-values.

Nice, isn't it? But this trick depends upon Frege's identification of truth-values as objects. Suppose we do not treat truth-values as objects but as complete entities of a different sort. Then we need some other notation for extensions, say, '$\hat{x}(Fx)$', and a new Law governing that

[23] Frege defines ordered pairs in terms of value-ranges, so value-ranges are not sets of ordered pairs. But pretend they are. Then the value-range of $\xi + 2$ is the set of ordered pairs $\{<\epsilon, \epsilon + 2>\}$. The value-range of $\epsilon + y$ is the set of ordered pairs $\{<\epsilon, \epsilon + y>\}$. The value-range of the function $\dot{\epsilon}(\epsilon + \eta)$ is the set of ordered pairs $\{<\alpha, \dot{\epsilon}(\epsilon + \alpha)>\}$, or $\{<\alpha, \{<\epsilon, \epsilon + \alpha>\}>\}$, which we might then identify with the set of ordered triples: $\{<\alpha, \epsilon, \epsilon + \alpha>\}$.

[24] Frege does not prove this result, since he does not need it in this form. Theorems 2 and 3 do the necessary work.

notation, say:[25]

$$\hat{x}(Fx) = \hat{x}(Gx) \equiv \forall x(Fx \equiv Gx)$$

Then the 'double extension' term "$\hat{y}\hat{x}(x < y)$", which one might have supposed would denote the extension of the relation $\xi < \eta$, is not even well-formed. Extension terms are formed by prefixing "$\hat{y}$", say, to a one-place *predicate*, but "$\hat{x}(x < \eta)$" is a functional expression, not a predicate: It denotes a function from objects to extensions.

The ugly workaround is to take the extension of a two-place predicate to be the value-range of a certain function from objects to extensions, that is: $\grave{\alpha}\hat{x}(x < \alpha)$. But there is more elegant solution. Select two arbitrary objects, which we shall denote "$\bot$" and "$\top$", and suppose we can show that, for any concept F, there is a function whose value is $\top$ for objects that fall under F and $\bot$ for objects that do not.[26] We call this function the concept's *characteristic function*.[27] Then the extension of a concept may be taken to be the value-range of its characteristic function. It is easy to see that, so defined, extensions satisfy the law governing them displayed above.

In fact, one can almost get by with nothing but characteristic functions. Instead of a relation of identity, for example, we could make use only of its characteristic function. An expression like "$2 + 2 = 4$" would then be a name of $\top$, not a sentence. And we can do the same with the logical constants: So "$(2 + 2 = 4) \wedge (1 + 1 = 3)$" is a name of $\bot$. To be able to form sentences, and so make assertions, we would need to have at least one real predicate in the language. The most natural choice for such a predicate would be one which meant: ξ is identical with the True. This, of course, is essentially Frege's horizontal:[28] In *Grundgesetze*, $-\!\!-\xi$ is a function whose value is the True for the True as argument, and the False otherwise.

But if we have come this far, we might well go a step farther and

[25] Here "$\equiv$" is the distinct notion of identity for truth-values.

[26] One way to do this is to use a description operator. Frege has such an operator in the formal system of *Grundgesetze*, but it is applied to value-range terms, not to predicates.

[27] Nowadays, the characteristic function associated with a set S is defined as:

$$\phi_S(x) = \begin{cases} 1, & \text{if } x \in S \\ 0, & \text{if } x \notin S \end{cases}$$

We have been unable to identify a clear antecedent of this notion prior to Frege's work, but do not know that there is not one.

[28] As we shall see in the next section, Frege did at one time hold that his begriffsschrift had only one real predicate.

identify concepts with their characteristic functions. Once the identification is made, the truth-values simply become the objects $\top$ and $\bot$, in terms of which the characteristic functions are defined, and "$2+2=4$" is both a name of $\top$ *and* a sentence.. Nothing could be more natural, mathematically speaking. Treating sentences as of the same logical type as proper names thus has substantial technical advantages in the context of Frege's system. And once one sees that it amounts simply to identifying concepts with their characteristic functions, it should not seem all that perplexing.

If the truth-values are objects, however, then the question arises which objects they are, in particular, whether they might be objects we already know by other names. Frege addresses this question in section 10 of *Grundgesetze*.[29] He first argues that the stipulations he has made about the references of the primitive expressions of his language do not determine which objects the truth-values are. More precisely, he argues that those stipulations do not determine whether the truth-values are value-ranges and, if so, of which functions. Frege then stipulates that the truth-values are to be identified with their own unit classes: the True is identified with the value-range of the concept that maps only the True to the True; the False, with that of the concept that maps only the False to the True. Burge (2005) has argued that this particular identification was required by Frege's other views, but we do not find his arguments convincing, in part because Frege seems to feel no need at all to justify the identification. Rather, Frege writes as if any stipulation at all will do. And if the interpretation given above is correct—if the thesis that truth-values are objects is really just an identification of concepts with their characteristic functions—then we can see why Frege feels free to stipulate which objects the truth-values are, so long as they are distinct. Why Frege thinks he needs to make any such stipluation is itself an interesting question, one far too large to discuss here. But making it the way he does must ultimately lead to disaster, since it obviously depends upon the inconsistent Basic Law V.

5 The Regress Argument: Why Truth is Not a Predicate

Much of the existing discussion of Frege's views on truth focuses on an argument he gives for the conclusion that truth is indefinable. This ar-

[29] Section 10 is not really about the truth-values but about an issue concerning value-ranges. There is an extensive literature on it.

gument, which has come to be known as the 'regress argument', appears in at least two places: the late essay, "Thoughts" (Tht, op. 60), written in 1917 or so, and an unfinished essay, "Logic", which the editors of the *Nachlass* date to 1897. Here is the argument as it occurs in "Logic":

> Now it would be futile to employ a definition in order to make it clearer what is to be understood by "true". If, for example, we wished to say that "an idea is true if it agrees with reality", nothing would have been achieved, since in order to apply this definition we should have to decide whether some idea or other did agree with reality. Thus we should have to presuppose the very thing that is being defined. The same would hold good of any definition of the form "A is true" if and only if it has such-and-such properties or stands in such-and-such a relation to such-and-such a thing'. In each case in hand it would always come back to the question whether it is true that A has such-and-such properties, stands in such-and-such a relation to such-and-such a thing. Truth is obviously something so primitive and simple that it is not possible to reduce it to anything still simpler. (Log97, pp. 128–9)

This argument is extremely puzzling. Some commentators have found in it an argument that there is no real property of truth at all (Ricketts, 1986; Kemp, 1995). But one would need a very strong argument indeed for this sort of claim, since there are so many places in Frege's writings where he seems to make serious use of semantic notions like reference and truth (Heck, 2010; Heck, 2012, Ch. 2). And, for the sorts of reasons given by Stanley (1996), Tappenden (1997), and Sullivan (2005), we do not think a strong argument has been given.

Perhaps what is most puzzling about the regress argument is that its first part contains what looks like a non-sequitur. Where does the question "whether some idea or other [does] agree with reality" presuppose the notion of truth? The later parts of the argument read differently. There Frege insists that the real question must be whether it is *true* that the idea agrees with reality. But that just seems gratuitous. Surely it is possible to ask whether Bob is home without asking whether it is true that Bob is home, let alone whether it is true that it is true that Bob is home, and so on and so forth, *ad infinitum*.

The key to Frege's thinking is revealed by remarks that follow the regress argument proper.

> What, in the first place, distinguishes ["true"] from all other predicates is that predicating it is always included in predicating anything whatever. If I assert that the sum of 2 and 3 is 5, then I thereby assert that it is true that 2 and 3 make 5. So I assert that it is true that my idea of Cologne cathedral agrees with reality, if I assert that it agrees with reality. Therefore, it is really by using the form of an assertoric sentence that we assert truth, and to do this we do not need the word "true". (Log97, p. 129)

What lies at the core of the regress argument, then, is the idea that every assertion is an assertion of truth; every judgement, a judgement of truth.

Frege makes this sort of claim in many places, for example, in "On Sense and Reference", where he writes: "A judgement, for me, is not the mere grasping of a thought, but the admission of its truth" (SM, op. 34, note). This sort of perspective is critical to Frege's larger conception of logic. Frege frequently emphasizes that his goal was "not. . . to present an abstract logic in formulas, but to express a content through written symbols in a more precise and perspicuous way than is possible with words" (AimCN, pp. 90–1; see also Geo1; Geo2). More precisely, Frege's logic was to be one we can actually use in reasoning, that is, in proving theorems, where theorems are true contents. So logic, in that sense, issues in judgements, in 'admissions of truth'.

The importance of this idea for the regress argument is clearest in an earlier presentation of essentially the same line of thought, again in "On Sense and Reference":

> One might be tempted to regard the relation of the thought to the True not as that of sense to reference, but rather as that of subject to predicate. One can indeed say: "The thought that 5 is a prime number is true". But closer examination shows that nothing more has been said than in the simple sentence "5 is a prime number". The truth claim arises in each case from the form of the assertoric sentence, and when the latter lacks its usual force, e.g., in the mouth of an actor upon the stage, even the sentence "The thought that 5 is a prime number is true" contains only a thought, and indeed the same thought as the simple "5 is a prime number". It follows that the relation of the thought to the True may not be compared with that of subject to predicate. (SM, op. 34)

Here, Frege is arguing that predication of truth is inadequate for assertion. Rather, Frege says again, the act of assertion is effected by using a sentence of a certain form.[30] But perhaps what is most noteworthy is the context in which this argument occurs. Frege is arguing against the view that truth is a property of thoughts because he sees it as a competitor to his view that "the relation of the thought to the True [is] that of sense to reference".

So we can reconstruct the regress argument as follows. Suppose we do think of truth as a property of thoughts (or propositions, or sentences, or what have you). Then the idea that judgement is admission of a thought's truth becomes the idea that judging is predicating truth of a thought. What the regress argument shows is that this cannot be right. Predication, in this sense, is a sort of judgement: To predicate truth of the thought that p is just to judge that the thought that p is true, that is, to judge that it is true that p. But then, to judge that it is true that p is to predicate truth of the thought that it is true that p, that is, to judge that it is true that it is true that p. The regress is vicious, since the sense in which judgement *is* predication of truth was meant to be constitutive.[31] So it is not just, as Frege puts it in "Logic", that, to "assert truth, . . . we do not need the word 'true'". Rather, the right conclusion to draw is the one from "On Sense and Reference": the word "true" *cannot* be used to assert truth.

What the regress argument shows, then, is that, while judgement does in some sense involve the acceptance of thoughts as true, to judge is not to predicate truth of a thought. We can therefore see why, as mentioned earlier, some commentators have been tempted to read Frege's argument as showing that there is no such property of thoughts as truth. That is almost its conclusion. The real conclusion, however, is that truth is not *fundamentally* a property of thoughts: The role truth plays in judgement—what we gesture at when we say that judgement involves recognition of a thought as true—is more basic and direct than that. To secure that role, Frege claims, we must instead conceive of the relation between a thought and its truth-value as that of sense to reference. But that does not prevent there from being a property had by all and only

[30] Davidson (1984) famously argues that the assertoric act is not conventional in the sense Frege seems to think it is. We think it unclear whether Frege is committed to such a view. His main point is the negative one: that predication of truth is inadequate for assertion.

[31] It would not be vicious if the claim were, say, merely one about the commitments one incurs by making a judgement, as Dummett (1981, Ch. 13) makes clear.

those thoughts that refer to the True. Indeed, we have just said which property it is.

Further confirmation of this interpretation of the regress argument emerges if we ask who its target is. Who ever thought that judgement is predicating truth of a thought? Frege himself! Consider the following passage from *Begriffsschrift*:

> We can imagine a language in which the proposition "Archimedes perished at the capture of Syracuse" would be expressed thus: "The violent death of Archimedes at the capture of Syracuse is a fact". To be sure, one can distinguish between subject and predicate here, too, if one wishes to do so, but the subject contains the whole content, and the predicate serves only to turn the content into a judgement. *Such a language would have only a single predicate for all judgements, namely, "is a fact". . . . Our begriffsschrift is a language of this sort, and in it the sign* $\vdash$ *is the common predicate for all judgements*. (Bg, §3, emphasis in original)

These remarks come at the conclusion of Frege's explanation of why the "distinction between subject and predicate does not occur in [his] way of representing a judgement" (Bg, §3). It is tempting, therefore, to regard them as but a grudging concession to tradition. But Frege emphasizes the final sentence of the quoted passage, and this sort of emphasis is used consistently in Part I of *Begriffsschrift* when Frege is articulating the central features of his new conception of logic. Frege is saying here, quite explicitly, that his begriffsschrift is a language in which there is only one predicate, the assertion-sign. What is most striking is Frege's remark that "the predicate ['is a fact'] serves. . . to turn the content into a judgement". This looks like an explicit claim that assertion is achieved through the predication of facthood.[32]

Frege thus held, throughout his career, that there is an intimate relationship between judgement and truth. But his original conception of this relation—that judgement is predication of truth—is unsatisfactory, because it falls to the regress argument. His mature view is that the relation between a sentence and its truth-value is to be modeled on the

[32] As Proops (1997) points out, Wittgenstein ascribes this view to Frege: "The verb of a proposition is not 'is true' or 'is false', as Frege thought; rather, that which 'is true' must already contain the verb" (Wittgenstein, 1961, 4.063; see also Wittgenstein, 1979a, pp. 93, 100). Proops was also the first to notice Frege's commitment to this view in *Begriffsschrift*.

relation between a name and its bearer. To judge that 5 is prime is thus not to predicate truth of the thought that 5 is prime. It is to entertain the thought that 5 is prime in an attempt to refer to the True,[33] thus taking "the step from the level of thoughts to the level of reference" (SM, op. 34). This, we think, is a promising and underappreciated idea, quite at odds with the way philosophers nowadays tend to think about truth. But we could hardly develop it here, even if we knew how to do so.[34]

The Truth-Value Thesis is thus not just a semantical or logical doctrine. It is Frege's attempt to explicate the important but maddeningly difficult idea that judgement (belief, assertion) constitutively aims at the truth.

6 Closing: Frege and Deflationism

In closing, we want to say a few words about another issue related to the regress argument, namely, whether Frege was a deflationist. The question here is not whether the regress argument itself has such a conclusion. As we have said, we reject such interpretations. But some, such as Horwich (1990, p. 39), have a much simpler reason for labeling Frege a deflationist, namely, that he claims that the sentence "The thought that five is prime is true" has the very same sense as "Five is prime" itself (SM, op. 34).[35]

The overall tenor of our discussion should have made it clear both that and why Frege is not a deflationist. Perhaps Frege did think that the predicate "is true" of natural language was redundant when applied to a thought explicitly identified by a clause expressing it.[36] But even so, Frege's view, as we saw in the last section, was emphatically that this is not truth as logic knows it. For logic, the True is one of the

[33] The attempting is critical. One will *in fact* refer to the True even if one utters "5 is prime" as the antecedent of a conditional.

[34] Textor (2010) and Jarvis (2012) develop related ideas.

[35] See also development of this argument by Kemp (1998) and the reply by Heck (2002).

[36] The depth of Frege's commitment to this view is not clear, however. There are several places in Frege's writings where he identifies the senses of sentences when one might have thought he should regard them as distinct (Heck and May, 2010, §5). Our view is that Frege does so because he tends to conflate a sufficient condition for difference of sense—that it should be possible to believe one but not the other—with a necessary condition—that it should, in some more practical sense, be possible to believe one but not the other. Frege gives no argument for the necessary condition, which has no real plausibility and plays no significant role in his philosophy. It is the sufficient condition that does the actual work.

two truth-values, and the truth-values are the referents of sentences. Logic would therefore be concerned with the True even if there were no truth-predicate as, indeed, there is not in Frege's logic.[37]

Moreover, Frege clearly would not agree with the common deflationist thesis that there is nothing substantial to be said about what it is for a thought to be true. For Frege, the thought that five is prime has a structure corresponding to that of the sentence "Five is prime" (CT, op. 36). Roughly, the thought that five is prime is composed of the sense of "five" and the sense of the predicate "is prime". These, in turn, determine references: the number five and the concept *is prime*, respectively. These then compose *via* function-application, so as to determine that the thought that five is prime has as its referent the True (Heck and May, 2010). This is a very long way from the view that "[t]he entire conceptual and theoretical role of truth may be explained" in terms of the assumption that it is true that five is prime if, and only if, five is prime (Horwich, 1990, p. 39). It is, rather, the very birth of semantics.

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[37] The horizontal is not a truth-predicate, any more than "$\xi = \forall x(x = x)$" is.

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