A Note on the Major Results of *Begriffsschrift*

Richard G. Heck, Jnr.

In this note, I shall explain and outline Frege's proofs of the two major results of *Begriffsschrift* (Frege, 1967), Part III, Theorems 98 and 133.[1] To do so, we need some notation. Following Frege's definition (69) in §24, we define when a concept $F\xi$ is *hereditary* in the *R*-series:

$$Her_{\alpha\epsilon\xi}(R\alpha\epsilon, F\xi) \equiv \forall x(Fx \rightarrow \forall y(Rxy \rightarrow Fy))$$

Thus, $F\xi$ is hereditary in the f-series if, and only if, every object to which an *F* bears *R* is *F*.[2]

We now define, following definition (76) in §26, when an object *b* follows after an object *a* in the *R*-series:

$$\overset{*}{_{\alpha\epsilon}}(R\alpha\epsilon, a, b) \equiv \forall F[Her_{\alpha\epsilon\xi}(R\alpha\epsilon, F\xi) \wedge \forall x(Rax \rightarrow Fx) \rightarrow Fb]$$

Thus, *b* follows after *a* in the *R*-series if, and only if, *b* falls under every concept (1) which is hereditary in the *R*-series and (2) under which every object to which a bears *R* falls. Note that the occurence of the second-order quantifier in this definition is essential: The ancestral cannot be defined in first-order logic.

We now define when an object *b* belongs to the *R*-sequence beginning with *a*, following definition (99) in §29:

$$\overset{*=}{_{\alpha\epsilon}}(R\alpha\epsilon, a, b) \equiv \overset{*}{_{\alpha\epsilon}}(R\alpha\epsilon, a, b) \vee a = b$$

Thus, *b* belongs to the *R*-sequence beginning with *a* if, and only if, *b* follows after *a* in the *R*-series or *b* is identical with *a*.

As we have just defined * and $^{*=}$, they are what Frege would have called 'three-place relations of mixed level': They are relations with three arguments, one of which is itself a two-place relation and the other two of which are objects. The expressions '*' and '$^{*=}$' thus have something in common with quantifiers: They bind the argument-places of the two-place relational expression that occurs in their first argument-place. That is why the variables 'α' and 'ϵ' appear where they do: The ones appearing as subscripts are analogous to the first '*x*' that occurs in '$\forall x(Fx)$'; the ones appearing in the argument-places of '*R*' are analogous to the second.

It is often more useful to think of * and $^{*=}$ as being operators that transform one relation into another relation: functions from relations to relations. That the target relations exist is assured by

[1] Thanks to George Boolos, notes of whose I have used in the preparation of these notes. The interested reader is invited to consult (Boolos, 1998) for further discussion.

[2] Note the use of the *bound variables* 'α', 'ϵ', and 'ξ'. '*Her*' is like a quantifier in so far as it binds the variable-places of the two-place predicate '$f\xi\eta$' and the one-place predicate '$F\xi$'. This is important, for without some such notation, as Frege in effect mentions, one would be quite unable to disambiguate certain sorts of sentences. However, when there is no danger of confusion, we shall later omit the bound variables.

the following two instances of the comprehension axiom of second-order logic:[3]

$$\exists Q \forall x \forall y [Qxy \equiv {}^{*}_{\alpha\epsilon}(R\alpha\epsilon, x, y)]$$

$$\exists Q \forall x \forall y [Qxy \equiv {}^{*=}_{\alpha\epsilon}(R\alpha\epsilon, x, y)]$$

We may therefore make the following definitions:

$$R^{*}(a, b) \equiv {}^{*}_{\alpha\epsilon}(R\alpha\epsilon, a, b)$$

$$R^{*=}(a, b) \equiv {}^{*=}_{\alpha\epsilon}(R\alpha\epsilon, a, b)$$

We call $R^{*}\xi\eta$ the *strong ancestral* of $R\xi\eta$ and $R^{*=}\xi\eta$ its *weak ancestral*.
Finally, then, we define, following Frege at (115) in §31, when a relation, or "procedure", $R\xi\eta$ is "single-valued" [*eindeutig*] or, as we shall say, *functional*:

$$Func_{\alpha\epsilon}(R\alpha\epsilon) \equiv \forall x \forall y (Rxy \rightarrow \forall z (Rxz \rightarrow y = z))$$

Thus, a relation $R\xi\eta$ is functional if each object bears it to at most one other object.
The two central theorems of Part III of *Begriffsschrift* may now be stated as follows. The first is the *transitivity of the strong ancestral*:

$$(98)\ R^{*}ab \wedge R^{*}bc \rightarrow R^{*}ac$$

The second is the *connectedness of the strong ancestrals of functional relations*:

$$(133)\ Func(R) \wedge R^{*}ab \wedge R^{*}ac \rightarrow [R^{*}bc \vee R^{*=}cb]$$

In what follows, we shall discuss Frege's proofs of these two results.

1 Theorem 98

Henceforth, we shall drop the bound variables when it causes no confusion to do so. Moreover, we shall freely change variables in Frege's statements of his results when doing so facilitates the exposition. We shall also make free use of propositional equivalences whose proofs in the formal system of *Begriffsschrift* can be quite tortuous.
Recall that Theorem 98 is:

$$(98)\ R^{*}ab \wedge R^{*}bc \rightarrow R^{*}ac$$

Frege's proof of this theorem proceeds via a method of proof we may call *strong ancestral induction*. Suppose that we know that *b* follows after *a* in the *R*-series and we wish to prove that *b* falls under some concept *F*. We can prove this if we can show (1) that *F* is hereditary in the *R*-series and (2) that everything to which *y* bears *R* has *F*. For, by the definition of the strong ancestral, we have:

[3] The embedded formulae are Π^1_1, of course.

$$(77)\ R^*ab \wedge Her(R,F) \wedge \forall x(Rax \rightarrow Fx) \rightarrow Fb$$

However, instead of proceeding in this way, we may show, more strongly, (1) that *F* is hereditary in the *R*-series and (2) that *a* itself has *F*:

$$(81)\ R^*ab \wedge Her(R,F) \wedge Fa \rightarrow Fb$$

The reason for this is simple. If *F* is hereditary in the *R*-series, if *a* is *F*, and if Rax, then, by the definition of heredity, *x* too must be *F*:

$$(72)\ Her(R,F) \wedge Fa \wedge Rax \rightarrow Fx$$

Generalizing, then, if *a* is *F* and *F* is hereditary in the *R*-series, then everything to which *a* bears *R* must also be *F*. So, by (77), *b* must be *F* if it follows after *a* in the *R*-series. That suffices for the proof of (81).

If we re-write (98), then, we can see how Frege intends to proceed:

$$(98)\ R^*bc \rightarrow [R^*ab \rightarrow R^*ac]$$

We suppose that *c* follows after *b* in the *R*-series. What Frege intends then to show is that, if *b* falls under the complex concept $R^*a\xi$, then so must *c*. That is: Frege intends to show that, if *b* has the property of *following after a in the R-series*, then so must *c*. To prove this, he intends to use strong ancestral induction, in the form of (81). What he must show is thus that the property of following after *a* in the *R*-series is hereditary in the *R*-series, whence if *b* has it, and *c* follows *b*, *c* must also have it. What needs to be shown is thus:

$$(97)\ Her_{\alpha\epsilon\xi}(R\alpha\epsilon, R^*a\xi)$$

Note how the bound variables are used to make clear just what concept is being said to be hereditary.

The proof, as it has so far been outlined, relies crucially at this point upon the use of a rule of substitution or, equivalently, the comprehension axiom of second-order logic. To conclude (98) from (81) and (97), Frege must substitute '$R^*a\xi$' for '$F\xi$' in (81) to get:

$$(81')\ R^*bc \wedge Her_{\alpha\epsilon\xi}(R\alpha\epsilon, R^*a\xi) \wedge Rab \rightarrow R^*ac$$

Theorem (98) then follows trivially from (81') and (97), but the use of substitution is essential and should carefully be noted.

Frege thus wishes to prove (97). He derives it, by the definition of heredity, from Theorem 96:

$$(96)\ R^*ay \wedge Ryz \rightarrow R^*az$$

For, if we generalize (96), we get

$$(96)\ \forall y[R^*ay \rightarrow \forall z(Ryz \rightarrow R^*az)]$$

and this is just the definition of $R^*a\xi$'s being hereditary in the *R*-series. So (97) follows from (96).

How then are we to prove (96): to conclude, from the assumptions that *y* follows after *a* in the *R*-series and that Ryz, that *c* follows after *a* in the *R*-series? What we need to show, by the definition of the strong ancestral, is that *c* has every *R*-hereditary 'property'[4] that everything to which *a* bears *R* has. That is to say, Frege will prove

$$(88')\ R^*ay \wedge Ryz \rightarrow \forall F[\forall x(Rax \rightarrow Fa) \wedge Her(R,F) \rightarrow Fz]$$

whose consequent is just the definition of 'R^*az, whence (96) follows from (88′).

Now (88′) is just a generalization of

$$(88)\ R^*ay \wedge Ryz \rightarrow [\forall x(Rax \rightarrow Fa) \wedge Her(R,F) \rightarrow Fz]$$

which is equivalent to

$$(88)\ R^*ay \wedge Ryz \wedge \forall x(Rax \rightarrow Fa) \wedge Her(R,F) \rightarrow Fz$$

But again, if we could conclude, from the antecedent, that *Fy*, it would follow immediately that *Fz*, since *Ryz* and *F* is *R*-hereditary. That is, (87) follows truth-functionally from the following two propositions:

$$(85)\ R^*ay \wedge \forall x(Rax \rightarrow Fx) \wedge Her(R,F) \rightarrow Fy$$
$$(72)\ Her(R,F) \wedge Fy \wedge Ryz \rightarrow Fz$$

But now we are essentially done. For (85) follows immediately from the definition of the strong ancestral, and (72), it was already said above, is immediate from the definition of heredity.

Thus is the transitivity of the strong ancestral proven by Frege.

It is worth noting that Frege's proof here is needlessly complicated. It is, in particular, possible to prove Theorem 98 without relying on the rule of subtitution, that is, on the comprehension axiom. Instead of using strong ancestral induction to prove (98), we can use the method used to prove (96). To prove that R^*ac, *c* has every *R*-hereditary 'property' that every object to which *a* bears *R* has. So we need to prove

$$(a)\ R^*ab \wedge R^*bc \rightarrow [\forall x(Rax \rightarrow Fx) \wedge Her(R,F) \rightarrow Fc]$$

which, generalized, will imply (98). To see how to prove (a), we re-write it as

$$(b)\ R^*ab \wedge R^*bc \wedge \forall x(Rax \rightarrow Fx) \wedge Her(R,F) \rightarrow Fc$$

[4] By an 'f-hereditary property' (or, better, 'concept') we mean, of course, one which is hereditary in the f-series.

Now, if we knew that y had F, we should easily conclude that z had F, by weak ancestral induction, that is, Theorem (81). *But here no use of the rule of substitution is required.* That is: If we could prove

$$(c)\ R^*ab \wedge \forall x(Rax \rightarrow Fx) \wedge Her(R, F) \rightarrow Fb$$

then (*b*) would follow truth-functionally from this and

$$(81)\ R^*bc \wedge Her(R, F) \wedge Fb \rightarrow Fc$$

But (*c*) follows immediately from the definition of the strong ancestral: Cf. (77) above.

2 Theorem 133

Recall that Theorem 133 is

$$(133)\ Func(R) \wedge R^*ab \wedge R^*ac \rightarrow [R^*bc \vee R^{*=}cb]$$

Note that, by the definition of the weak ancestral, Theorem 133 is equivalent to

$$(133)\ Func(R) \wedge R^*ab \wedge R^*ac \rightarrow [R^*bc \vee b = c \vee R^*cb]$$

Theorem 133 is thus a ‘logical’ version of the arithmetical law of trichotomy: $\forall x \forall y \forall z[x < y \vee x = y \vee y < x]$.

To prove Theorem 133, Frege essentially uses strong ancestral induction, that is, (81). But he uses it in the following rather special form:

$$(83)\ Her_{\alpha\epsilon\xi}(R\alpha\epsilon, H\xi \vee G\xi) \wedge R^*xy \wedge Hx \rightarrow Hy \vee Gy$$

What (83) says is that, if being either *H* or *G* is *R*-hereditary, and if *y* follows after *x* in the *R*-series, and if *x* is itself *H*, then *y* must be either *H* or *G*. This follows quickly from (81). For, substituting ‘$H\xi \vee G\xi$’ for ‘$F\xi$’ in (81), we have:

$$(81')\ R^*yz \wedge Her(R\alpha\epsilon, H\xi \vee G\xi) \wedge (Hx \vee Gx) \rightarrow Hy \vee Gy$$

But (83) obviously follows truth-functionally from (81’).

Now, if we substitute ‘$R^*\xi c$’ for ‘$H\xi$’, ‘$R^{*=}c\xi$’ for ‘$G\xi$’, ‘*a*’ for ‘*x*’, and ‘*b*’ for ‘*y*’ in (83), we get

$$(83')\ Her(R\alpha\epsilon, R^*\xi c \vee R^{*=}c\xi) \wedge R^*ab \wedge R^*ac \rightarrow [R^*bc \vee R^{*=}cb]$$

And (133) will now follow from this if the *R*-heredity of $R^*\xi c \vee R^{*=}c\xi$ can be shown to follow from the functionality of $R\xi\eta$. That is: (133) follows from (83’) and

$$(131)\ Func(R) \to Her(R\alpha\epsilon, R^*\xi c \vee R^{*=}c\xi)$$

Theorem 131 is thus the main lemma of the proof.
Theorem 131 will follow, by the definition of heredity, from

$$(129)\ Func(R) \to \{[R^*yc \vee R^{*=}cy] \wedge Ryz \to [R^*zc \vee R^{*=}cz]\}$$

which is truth-functionally equivalent to

$$(129')\ Func(R) \wedge [R^*yc \vee R^{*=}cy] \wedge Ryz \to [R^*zc \vee R^{*=}cz]$$

This is to be proven by cases. More precisely, (129′) follows truth-functionally from these:

$$(111)\ R^{*=}cy \wedge Ryz \to R^*zc \vee R^{*=}cz$$
$$(126)\ Func(R) \wedge R^*yc \wedge Ryz \to R^*zc \vee R^{*=}cz$$

We discuss the easier Theorem 111 first.
It follows truth-functionally from

$$(108)\ R^{*=}cy \wedge Ryz \to R^{*=}cz$$

What (108) says is that, if *y* belongs to the *R*-series beginning with *c* and *Ryz*, then *z* too belongs to the *R*-series beginning with *c*. The proof of this basic fact about the weak ancestral is straightforward.
Since, by the definition of the weak ancestral,

$$(106)\ R^*cz \to R^{*=}cz$$

(108) follows from the stronger statement that, if *y* belongs to the *R*-series beginning with *c* and *Ryz*, then *z follows after c* in the *R*-series:

$$(102)\ R^{*=}cy \wedge Ryz \to R^*cz$$

By the definition of the weak ancestral, (102) is equivalent to

$$(102')\ [R^*cy \vee c = y] \wedge Ryz \to R^*cz$$

which may be proven by cases. One case is just (a substitution instance of) the previously mentioned

$$(96)\ R^*cy \wedge Ryz \to R^*cz$$

and the other is

$$(92)\ c = y \wedge Ryz \rightarrow R^{*}cz$$

But (92) follows, by the laws of identity, from

$$(91)\ Rcz \rightarrow R^{*}cz$$

To prove (91), we proceed in the same way as in the proof of (96). What we want to prove is

$$(d)\ Rcz \rightarrow [\forall x(Rcx \rightarrow Fx) \wedge Her(R, F) \rightarrow Fz]$$

from whose generalization and the definition of the strong ancestral (91) will follow. The proof of (*d*) is easy and is left as an exercise.

To complete the proof of Theorem 133, we thus need only to complete the proof of Theorem 126, which is, recall:

$$(126)\ Func(R) \wedge R^{*}yc \wedge Ryz \rightarrow R^{*}zc \vee R^{*=}cz$$

(126) follows truth-functionally from the following two lemmas:

$$(124)\ Func(R) \wedge R^{*}yc \wedge Ryz \rightarrow R^{*=}zc$$

$$(114)\ R^{*=}zc \rightarrow R^{*}zc \vee R^{*=}cz$$

(114) is left as an exercise for the reader.

It is worth reviewing the proof of Theorem 133 to this point. Let us call $R^{*}\xi c \vee R^{*=}c\xi$ the property of *being ancestrally related to c*. We wish to show that if *R* is functional and both *b* and *c* follow after *a* in the *R*-series, then *b* must be ancestrally related to *c*. To prove this, it is enough to show that, if *R* is functional, the property of being ancestrally related to *c* is *R*-hereditary (131). For then, by strong ancestral induction (and substitution!), if *R* is functional, if *a* is ancestrally related to *c*, and if *b* follows after *a* in the *R*-series, then *b* too must be ancestrally related to *c*. For, if *c* follows after *a* in the *R*-series, then *a* is, *a fortiori*, ancestrally related to *c*.

Now, to show that being ancestrally related to *c* is hereditary, we suppose that *R* is functional, that either $R^{*}xc$ or $R^{*=}cx$ and that Rxy. We must show that either $R^{*}yc$ or $R^{*=}cy$. There are then two cases. If $R^{*=}cy$—that is, if *y* belongs to the *R*-series beginning with *c*—then, if *Ryz*, surely *z* too belongs to the *R*-series beginning with *c* (108), whence it is certainly ancestrally related to *c* (111). The other, more difficult, case (126) is decided by Theorem 124, which is an important result in its own right.

Theorem 124 is to be proven, once again, by strong ancestral induction. Substituting '$R^{*=}c\xi$' for '$F\xi$', '*a*' for '*y*', and '*c*' for '*b*' in (77), we have:

$$(77)\ R^{*}yc \wedge Her(R\alpha\epsilon, R^{*=}z\xi) \wedge \forall x(Ryx \rightarrow R^{*=}zx) \rightarrow R^{*=}zc$$

(124) follows truth-functionally from this,

$$(109)\ Her(R\alpha\epsilon, R^{*=}z\xi)$$

and

$$(122')\ Func(R) \wedge Ryz \to \forall x(Ryx \to R^{*=}zx)$$

The remainder of the proof of (124) is not difficult. First, (122′) is just a generalization of

$$(122)\ Func(R) \wedge Ryz \wedge Ryx \to R^{*=}zx$$

which follows from

$$(112)\ z = a \to R^{*=}za$$

which is a trivial consequence of the definition of the weak ancestral, and

$$(120)\ Func(R) \wedge Ryz \wedge Rya \to z = a$$

which follows from the definition of functionality. Finally, (109) follows from

$$(108)\ R^{*=}zu \wedge Ruv \to R^{*=}zv$$

which was mentioned above, and the definition of heredity.
That, then, completes the proof of Theorem 133.

References

Boolos, G. (1998). Reading the *Begriffsschrift*. In *Logic, Logic, and Logic*, pages 155–170. Harvard University Press, Cambridge MA.

Frege, G. (1967). Begriffsschrift: A formula language modeled upon that of arithmetic, for pure thought. In van Heijenoort, J., editor, *From Frege to Godel: A Sourcebook in Mathematical Logic*, pages 5–82. Harvard University Press, Cambridge MA. Tr. by J. van Heijenoort.