TRUTH-THEORIES FOR FRAGMENTS OF PA

RICHARD G. HECK, JR.

The discussion here follows Petr Hájek and Pavel Pudlák, *Metamathematics of First-order Arithmetic* (Berlin: Springer-Verlag, 1993). See especially Ch.I.1.

1. *Some Definitions*

- An occurrence of a quantifier is said to be *bounded* if it, together with the formula in its scope, is of the form '$\exists x(x < n \wedge \phi)$' or '$\forall x(x < n \rightarrow \phi)$'.
- A formula is said to be Δ_0 (equivalently, Σ_0 or Π_0) if all occurrences of quantifiers within it are bounded.
- A formula is said to be Σ_{n+1} if it is of the form $\exists x.\phi$, where ϕ is Π_n; Π_{n+1}, if it is of the form $\forall x.\phi$, where ϕ is Σ_n.
- A formula ϕ is said to be Σ_n [Π_n] *in a theory T* if T proves that ϕ is equivalent to a Σ_n [Π_n] formula.
- Note that, according to these definitions, $\exists x\exists y.\phi(x, y)$, where $\phi(x, y)$ is Δ_0, is not Σ_1. But in any theory that admits a Δ_0 pairing function, and proves certain basic facts about it, this formula is provably equivalent to a Σ_1 formula, namely, to $\exists s \exists x < s \exists y < s(s = (x, y) and \phi(x, y))$, and so it is Σ_1 in such theories. Such 'contraction of quantifiers' is possible in all the theories we shall be discussing.
- A Σ_n [Π_n] formula ϕ is said to Δ_n in T if there is a Π_n [Σ_n] formula ψ such that T proves $\phi \equiv \psi$.
- Note that the very notion of a formula's being Δ_n is a relative one: relative to a background theory. One can speak of a formula's being Δ_n in an *absolute* sense and mean by this: Δ_n in (true) arithmetic.
- $I\Gamma$ is the theory containing as axioms all axioms of *Q* and induction for all formulae in the set Γ. So, e.g., $I\Sigma_1$ is *Q* plus induction for all Σ_1 formulas, etc.

This does not define what Δ_n is: It is *Q* plus all axioms of the form:

$$\forall x(\phi x \equiv \psi x) \rightarrow [\phi 0 and \forall x(\phi x \rightarrow \phi(Sx)) \rightarrow \forall x.\phi x]$$

for all Σ_n formulae ϕ and Π_n formulae ψ.

2. *Facts about* $I\Sigma_1$

Fact 2.1. *We can define notions of term, formula, proof, sequence, etc., in such a way that (i) all the defined notions are* Δ_1 *in* $I\Sigma_1$ *and (ii)* $I\Sigma_1$ *proves the basic facts about them, including schemata allowing proof by induction on the complexity of expressions, the length of proofs, etc.*

Proof. See Hájek and Pudlák for the details. For the most part, the proof simply consists in carefully recording the forms of the various notions defined. □

Fact 2.2. *Formulas* Δ_1 *in* $I\Sigma_1$ *are closed under connectives and bounded quantification. I.e., if* ϕ *and* ψ *are* Δ_1 *in* $I\Sigma_1$*, then* $\neg\varphi$, $\varphi \wedge \psi$, $(\exists x < n)\phi$, *and* $(\forall x < n)\phi$ *are* Δ_1 *in* $I\Sigma_1$.

Proof. The cases of connectives are easy. (Just compute the prenex formula, being careful to bring the quantifiers out in the right order.) To complete the proof, show by induction that, for every Δ_0 formula $\phi(x, y)$, $I\Sigma_1$ proves: $(\forall x < n)(\exists y)\phi(x, y) \rightarrow (\exists u)(\forall x < n)(\exists y < u)\phi(x, y)$. It follows that formulae Σ_1 in $I\Sigma_1$—that is, the set of formulae that are provably equivalent in $I\Sigma_1$ to a Σ_1 formula—are closed under bounded quantification. Now let ψ be Π_1. Then $\neg\psi$ is Σ_1 in $I\Sigma_1$, so $(\forall x < n)\neg\psi$ is Σ_1 in $I\Sigma_1$, too, by the above. But then $\neg(\forall x < n)\neg\psi$ is Π_1 in $I\Sigma_1$ and is logically equivalent to $(\exists x < n)\psi$. So the formulae that are Π_1 in $I\Sigma_1$ are also closed under bounded quantification. □

Fact 2.3. *In $I\Sigma_1$, we can define Δ_1 functions from other Δ_1 functions by composition and primitive recursion. It follows that all primitive recursive functions are defined by formulas Δ_1 in $I\Sigma_1$ (since the basic primitive recursive functions are obviously Δ_1 in $I\Sigma_1$). $I\Sigma_1$ will prove the basic facts about such functions.*

Proof. Again, see Hájek and Pudlák. □

Here is an interesting fact we shall not be using. Expand the language of arithmetic by adding a unary function symbol $exp(x)$, interpreted as 2^x. Add the following axioms to $I\Delta_0$:

$$exp(0) = 1$$
$$exp(Sn) = 2 \times exp(n)$$

Bounded formulae containing exp are called $\Delta_0(exp)$; those containing bounded quantifiers of the form $(\forall x < exp(y))$ are called $\Delta_0^{exp}(exp)$. If we allow induction over $\Delta_0(exp)$ formulae, we get a theory called $I\Delta_0(exp)$; over $\Delta_0^{exp}(exp)$ formulae, $I\Delta_0^{exp}(exp)$. $I\Delta_0(exp)$ proves all instances of induction for $\Delta_0^{exp}(exp)$ formulae, so these are really the same theory. Moreover, $I\Delta_0^{exp}(exp)$ interprets a version of hereditarily finite set theory in which separation is limited to $\Delta_0^{exp}(exp)$ formulae: So one can freely speak, to some extent at least, of finite sets in this theory. Moreover, $I\Sigma_1$ interprets $I\Delta_0^{exp}(exp)$: So one can freely speak of such finite sets in $I\Sigma_1$ as well. This makes formalization of certain definitions in $I\Sigma_1$ an easy matter.

Definition 2.4. An *evaluation* is a finite mapping from variables to natural numbers. (Strictly speaking, it's a sequence of ordered pairs satisfying certain conditions guaranteeing that it is well-defined: It can also be defined *via* the finite set theory just mentioned.—We will ignore this sort of detail henceforth.) Write $e(v)$ for the value that *e* assigns to the variable *v*.

e is an evaluation *for the term or formula u* if *e* assigns values to all free variables of *u*. (Write $Fv(u)$ for: the set of free variables of *u*.)

The above notions are Δ_1 in $I\Sigma_1$. (In fact, they are of lower complexity, but we shall not need that fact.)

Fact 2.5. *There is a formula $Val(t, e, x)$ which is Δ_1 in $I\Sigma_1$ and which represents: the term t denotes x with respect to the evaluation e. $I\Sigma_1$ proves its basic properties. That is, whenever e is an evaluation for t, $I\Sigma_1$ proves:*

(1) *If t is a term, then there is a unique x such that $Val(t, e, x)$. (Write $val(t, e)$ for $(\iota x)Val(t, e, x)$, where it exists.)*
(2) $val(‘0’, e, 0)$
(3) *If v is a variable,* $val(v, e) = e(v)$.
(4) $val(\ulcorner St \urcorner, e) = S(val(t, e))$
(5) $val(\ulcorner t + u \urcorner, e) = val(t, e) + val(u, e)$

(6) $val(\ulcorner t \times u \urcorner, e) = val(t, e) \times val(u, e)$

Proof. Exercise. (Hint: *Val* is primitive recursive.) □

Fact 2.6. *$I\Sigma_1$ defines satisfaction for Δ_0 formulas. That is, there is a formula $Sat_0(z, e)$ which is Δ_1 in $I\Sigma_1$ and for which $I\Sigma_1$ proves Tarski's conditions:*

(1) *$Sat_0(z, e) \rightarrow$ z is Δ_0 and e is an evaluation for z.*
(2) *If $z = \ulcorner t = u \urcorner$ then $Sat_0(z, e)$ iff $val(t, e) = val(u, e)$.*
(3) *If z is $\ulcorner \neg u \urcorner$, then $Sat_0(z, e)$ iff $\neg Sat_0(u, e)$.*
(4) *If z is $\ulcorner t \vee u \urcorner$, then $Sat_0(z, e)$ iff $Sat_0(t, e) \vee Sat_0(u, e)$.*
(5) *If z is $\ulcorner t \wedge u \urcorner$, then $Sat_0(z, e)$ iff $Sat_0(t, e) \wedge Sat_0(u, e)$.*
(6) *If z is $\ulcorner t \rightarrow u \urcorner$, then $Sat_0(z, e)$ iff $Sat_0(t, e) \rightarrow Sat_0(u, e)$.*
(7) *If z is $\ulcorner (\forall w < v) u \urcorner$, then $Sat_0(z, e)$ iff for every d satisfying the three conditions:*
  (a) d(w) is defined
  (b) if $t \neq w$ and $t \in Fv(u)$, then $d(t) = d(t)$
  (c) $d(w) \leq d(v)$
  $Sat_0(u, d)$.

Proof. See below. □

Of course, we may define $Tr_0(u)$ as: $\forall e Sat_0(z, e)$. Note that this is also Δ_1 in $I\Sigma_1$, since it is provably equivalent to: $\exists e Sat_0(z, e)$, in virtue of a theorem to be stated and proved below.

Fact 2.7. *For every n, there is a formula $Sat_{\Sigma,n}(z, e)$ [resp. $Sat_{\Pi,n}(z, e)$] which is Σ_n [resp. Π_n] in $I\Sigma_1$ such that $I\Sigma_1$ proves the Tarskian conditions for $Sat_{\Sigma,n}$ and for Σ_n formulae [resp. for $Sat_{\Pi,n}$ and for Π_n formulae]: That is, it proves analogues of (1)–(6) above and the usual conditions for the quantifiers.*

Proof. Assume we have defined Sat_0 as in 2.6, above, and $Sat_{\Sigma,m}$ and $Sat_{\Pi,m}$, for $m < n$. Now define $Sat_{\Sigma,n}(z, e)$ iff (i) *z* is Π_m for some $m < n$ and $Sat_{\Pi,m}(z, e)$ or (ii) *z* is Σ_m for some $m < n$ and $Sat_{\Sigma,m}(z, e)$ or (iii) *z* is Σ_n—i.e., is $(\exists v)\phi$ for ϕ Π_{n-1}—and *e* is an evaluation for *z* and for some evaluation *d* agreeing with *e* on $Fv(\phi)$, $Sat_{\Sigma,n}(z, d)$. Then clearly, $Sat_{\Sigma,n}$ is Σ_n in $I\Sigma_1$, and one can easily show that $I\Sigma_1$ proves Tarski's conditions for $Sat_{\Sigma,n}$.
$Sat_{\Pi,n}$ is defined similarly. □

Again, we may define $Tr_{\Sigma,n}(z)$ as: $\exists e(Sat_{\Sigma,n}(z, e))$. It follows that $Tr_{\Sigma,n}(z)$ is Σ_n in $I\Sigma_1$.

Corollary 2.8. *For every Σ_n formula $\phi(x_0, \ldots, x_n)$,*

$$I\Sigma_1 \vdash \phi(x_0, \ldots, x_n) \equiv Sat_{\Sigma,n}(\ulcorner \phi(x_0, \ldots, x_n) \urcorner, < x_0, \ldots, x_n >)$$

And for every Σ_n sentence ϕ,

$$I\Sigma_1 \vdash \phi \equiv Tr_{\Sigma,n}(\phi)$$

Proof. By induction on the complexity of expressions. □

Note that the corollary tells us that we can develop a truth-theory for Σ_n sentences not in $I\Sigma_n$ or $I\Sigma_{n+1}$, which one might have expected, but already in $I\Sigma_1$! This gives us a respect in which IΣ_1, though in obvious ways a weak system of arithmetic, is actually quite strong.

Fact 2.9. *In particular, there is a formula $Sat_{\Sigma,1}(z, e)$ that is Σ_1 in $I\Sigma_1$ such that $I\Sigma_1$ proves the Tarskian conditions for $Sat_{\Sigma,1}(z, e)$ and proves all T-sentences for Σ_1 sentences and formulae.*

Why doesn't the liar paradox arise? By diagonalization, there is indeed a formula λ such that:

$$I\Sigma_1 \vdash \lambda \equiv \neg Tr_{\Sigma,1}(\lambda)$$

Since λ is $I\Sigma_1$-provably equivalent to the negation of a Σ_1 formula, it is Π_1 in $I\Sigma_1$: But using $Tr_{\Sigma,1}$, we can only prove T-sentences for Σ_1 formulae. Indeed, $Tr_{\Sigma,1}(\phi)$ is trivially false if ϕ is not Σ_1. If λ *were* Σ_1 in $I\Sigma_1$, then, we *would* get the liar paradox. But that only shows that λ is *not* Σ_1. Similar remarks apply to the other truth-predicates we have defined.

Finally, a couple of consequences worth knowing.

Corollary 2.10. *Using the formulae $Sat_{\Sigma,n}$, we can introduce notation for Σ_n sets into $I\Sigma_1$ and so code quantification over such sets, for any n, as follows. Say that x is a Σ_n-set if x is the Gödel number of a Σ_n formula whose only free variable is x_0. Define $y \in_{\Sigma,n} x$ iff $Sat_{\Sigma,n}(x, e)$, where e is the evaluation that assigns y to x_0 and makes no other assignments. By the above, we have:*

$$y \in_{\Sigma,n} \ulcorner\phi(x_0)\urcorner \textit{ iff } Sat_{\Sigma,n}(\ulcorner\phi(x_0)\urcorner, e) \textit{ iff } \phi(y)$$

Note that $\in_{\Sigma,n}$ is Σ_n in $I\Sigma_1$, so induction over formulae containing it will be possible in $I\Sigma_n$.[1]

Appeal to this fact will make our work below much easier.

Corollary 2.11. *$I\Sigma_n$ is finitely axiomatized.*

Proof. Basically, consider:

$$\forall x(x \text{ is a } \Sigma_n \text{ set } \wedge 0 \in_{\Sigma,n} x \wedge \forall y(y \in_{\Sigma,n} x \rightarrow Sy \in_{\Sigma,n} x \rightarrow \forall y(y \in_{\Sigma,n} x))$$

The open part of this is an instance of Σ_n induction, so is provable in $I\Sigma_n$. But by 2.10, every instance of Σ_n induction follows from this, too. The finite axiomatization then consists of enough of $I\Sigma_1$ to define $\in_{\Sigma,n}$ and prove that it has the properties mentioned above—since these are a finite collection of results, only finitely much of $I\Sigma_1$ is needed for their proofs—plus the sentence just mentioned. □

Although this is hardly an immediate consequence, one can show, given what has been said so far, that $I\Sigma_{n+1}$ proves $Con(I\Sigma_n)$. It follows that PA proves $Con(I\Sigma_n)$, for each n. (For a proof, see Ch.I.4 of Hájek and Pudlák.) Hence, for any finite subset Γ of axioms of PA, $PA \vdash Con(\Gamma)$. PA is thus 'reflexive', as it is sometimes put.

This gets us as close to paradox as one can get without getting paradox: The union of all the $I\Sigma_n$ just is PA! So PA cannot prove: $\forall n Con(I\Sigma_n)$—this is meaningful, and true—and so we have an example of a respect in which PA is ω-incomplete.

3. Defining Δ_0-Truth in $I\Sigma_1$

Definition. An evaluation e is a *partial valuation* for variables with Gödel numbers less than or equal to $g$and values less than or equal to n, if e is a finite mapping[2] from $\{\leq \times \leq n\}$ to $\mathbb{N}$. (Note that e need not be defined for all variables with Gödel numbers less than g.) Write: $PV(e, g, n)$ for this.

[1]Is there a natural formulation of the set theory this allows us to interpret in $I\Sigma_n$? The set theory will have an axiom of infinity (since $x = x$ is obviously Σ_n, for all *n*); it follows that it does not have power set. We will have separation for Σ_n formulae. So is the set theory *Z* without power set, with separation so restricted? I do not know the answer to this question.

[2]Finite mappings are sequences of ordered pairs satisfying the condition that each 'argument' is assigned at most one 'value'. The notion is Δ_1 in $I\Sigma_1$.

This will work because we are defining satisfaction for formulae containing *bounded* quantifiers. Consider, for example, a formula of the form $(\forall w < v)u$, with u closed. We want to say when this is satisfied by a evaluation e. Any such evaluation must assign a value to v, say, n. So, since the quantifier is bounded, we need consider only valuations that assign some value *less than* n to w. Moreover, we can, of course, choose the Gödel numbering so that the Gödel number of a part of a formula is always less than the Gödel number of that formula. This fact makes it possible to use the notion of partial satisfaction in defining truth for Δ_0 formulae.

Definition. Let $s(z, e)$ be a finite mapping from formulas and evaluations to truth-values ($\in \{0, 1\}$). We say that s is a *partial satisfaction* for Δ_0 formulas with Gödel numbers less than g and values less than n (write: $PSat_0(s, g, n)$), if:

(1) s is defined for all pairs (z, e) such that $z \leq g$ and z is the Gödel number of a Δ_0 formula and e is such that $PV(e, g, n)$ and e is an evaluation for z;

and whenever s is defined for the things to be mentioned,

(2) $s(\ulcorner u = v \urcorner, e) = 1$ iff $val(u, e) = val(v, e)$;
(3) $s(\ulcorner \neg u \urcorner, e) = 1$ iff $s(u, e) = 0$;
(4) $s(\ulcorner u \wedge v \urcorner, e) = 1$ iff $s(u, e) = 1 \wedge s(v, e) = 1$;
(5) $s(\ulcorner (\forall w < v)u \urcorner, e) = 1$ iff, for every d such that $PV(d, g, n)$ which agrees with e on $Fv(u)$, except possibly on w, and which is such that $d(w)$ is defined[3] and $< d(v)$, we have $s(u, d) = 1$.

In essence, a partial satisfaction satisfies Tarski's conditions wherever it is defined.

Lemma 3.1. *$PSat_0$ is Δ_1 in $I\Sigma_1$.*

Proof. Check the definition. □

Definition. $Sat_0(z, e)$ iff $\exists s \exists g \exists n[PSat_0(s, g, n)) \wedge s(z, e) = 1]$

So an evaluation satisfies a formula if there is some partial satisfaction with respect to which the evaluation satisfies the formula.

Lemma 3.2. *$I\Sigma_1$ proves: Suppose $PSat_0(s, g, n)$. If e and e' are evaluations for z, $PV(e, g, n)$ and $PV(e', g, n)$, and e and e' agree on all free variables contained in z (written: $e\ {}_z e'$), then $s(z, e) = s(z, e')$.*

Note: $e\ {}_z e'$ is Δ_1 in $I\Sigma_1$. (The newly introduced quantifiers may be bounded by the greater of e and e'.)

Claim. If t is a term and $e\ {}_z e'$, then $val(t, e) = val(t, e')$. (Exercise!)

Proof. (of the Lemma) By induction on the complexity of formulas, i.e., on that of z. (Note that, by assumption, $s(z, e)$ is defined.)
For atomic statements: $s(\ulcorner u = v \urcorner, e) = 1$ iff $val(u, e) = val(v, e)$, by definition, iff $val(u, e') = val(v, e')$, by the claim, iff $s(\ulcorner u = v \urcorner, e') = 1$, by definition. For negation: $s(\ulcorner \neg u \urcorner, e) = 1 - s(u, e)$, by definition, $= 1 - s(u, e')$, by the induction hypothesis, $= s(\ulcorner \neg u \urcorner, e')$, by definition.

[3]The bit about $d(w)$ being defined is due to the fact that partial valuations need not be defined for all variables. It is more convenient, as one can see by looking at the proofs below, to allow this.

For conjunction: $s(\ulcorner u \wedge v \urcorner, e) = 1$ iff $s(u,e) = 1 \wedge s(v,e) = 1$, by definition, iff $s(u,e') = 1 \wedge s(v,e') = 1$, by the induction hypothesis, iff $s(\ulcorner u \wedge v \urcorner, e') = 1$, by definition.

Bounded quantifiers are only slightly harder. Let z be $(\forall w < v)u$. Say that $d\,{}^*e$ if d agrees with e on $Fv(u)$, except possibly on w, and $d(w)$ is defined and $< d(v)$. Then, by definition, $s(z,e) = 1$ iff, for all d such that $PV(d,g,n)$ and $d\,{}^*e$, $s(u,d) = 1$; similarly, $s(z,e') = 1$ iff, for all d such that $PV(d,g,n)$ and $d\,{}^*e'$, $s(u,d) = 1$. But now, $d\,{}^*e$ iff $d\,{}^*e'$, since $e\,{}_z e'$, and we are done. □

Lemma 3.3. *$I\Sigma_1$ proves: If $PSat_0(s,g,n)$ and $PSat_0(s',g',n')$, then if $s(z,e)$ and $s'(z,e)$ are both defined, they are equal.*

Proof. By induction on the complexity of the formula z. We prove: If $PV(e, min(g,g'), min(n,n'))$, then $s(z,e) = s'(z,e)$. (These are necessarily defined, since both $PV(e,g,n)$ and $PV(e,g',n')$.) The induction is legitimate because of 3.1—and because Δ_1 formulas are closed under connectives.

All cases are easy except bounded quantifiers. Let z be the formula: $(\forall w < v)u$; let e be as said. Note that, if $s(z,e)$ is defined, then $e(v) \leq n$. If $s(z,e) = 0$, then (using notation from above), for some $d\,{}^*e$, $s(u,d) = 0$. Let d' be the restriction of d to variables in u. Then $PV(d', min(g,g'), min(n,n')$, so $s(u,d') = 0$, by 3.2, and so, by the induction hypothesis, $s'(u,d') = 0$, whence $s'(z,e) = 0$, again by 3.2.

So suppose $s(z,e) = 1$. Then for all d such that $PV(d,g,n)$ and $d\,{}^*e$, we have $s(u,d) = 1$. Suppose $d'\,{}^*e$ and $PV(d',g',n')$. Let d'' be the restriction of d' to variables in u. Since $d''(w) = d'(w) < d'(v) = e(v) \leq n$, we have that $PV(d'', min(g,g'), min(n,n')$; moreover, $d''\,{}^*e$. So $s(u,d'') = 1$, by 3.2, and by the induction hypothesis, $s'(u,d'') = 1$. But then $s'(u,d') = 1$, by 3.2, and so $s'(z,e) = 1$. □

The foregoing shows that partial satisfactions never conflict: If one tells us that e satisfies z, then all do. This is a kind of uniqueness claim. What we now need to prove is the corresponding existence claim: That for any g and n, there will be a partial satisfaction s that is defined for all pairs (z,e) such that $z \leq g$ is the Gödel number of a Δ_0 formula and e is a partial valuation for variables with Gödel numbers less than or equal to $g$and values less than or equal to n.

Lemma 3.4. *$I\Sigma_1$ proves: For all g and n, there is an s such that: $PSat_0(s,g,n)$.*

Proof. By induction on g in $\exists s PSat_0(s,g,n)$, with n as a parameter. (Note that this is Σ_1, so the induction is legitimate.) The basis case, $g = 0$, is trivial. So suppose $PSat_0(s,g,n)$ and consider $h = Sg$. If h is neither (the Gödel number of) a Δ_0 formula nor (that of) a variable, then $PSat_0(s,h,n)$, trivially. If h is a variable, then—while evaluations may make assignments to h—the value of h will be irrelevant, since h cannot occur in any formula whose Gödel number is less than h. So given an evaluation e, let e' be e restricted to variables other than h and put $s'(u,e) = s(u,e')$. It is then easy to see that $PSat_0(s',h,n)$.

Suppose now that h is (the Gödel number of) a Δ_0 formula. Since h is not a variable, the evaluations will assign values only to variables $\leq g$. We must extend $s(u,e)$ to some $s'(u,e)$. In any such case, if $u < h$, then $s(u,e)$ is already defined, so set $s'(u,e) = s(u,e)$; fix the value of $s'(h,e)$ according to the definition of $PSat_0$ given above: That is, if h is (the Gödel number of) $(\forall w < v)t$, then set $s'(h,e) = 1$ if, for every d such that $PV(d,h,n)$—and these are only d such that $PV(d,g,n)$, since $h = Sg$ and h is not (the Gödel number of) a variable—which agrees with e on $Fv(t)$, except possibly on w, and which is such that $d(w)$ is defined and $< d(v)$, we have $s(u,d) = 1$; otherwise, set $s'(h,e) = 0$. It is again easy to see that $PSat_0(s',h,n)$. □

Lemma 3.5. *Sat_0 is Δ_1 in $I\Sigma_1$.*

Proof. $Sat_0(z, e)$ is obviously Σ_1 in virtue of how it is defined. But by the preceding lemmas, it is equivalent in $I\Sigma_1$ to: $\forall s \forall g \forall n (PSat_0(s, g, n) \wedge s(z, e)$ is defined $\rightarrow s(z, e) = 1)$. And now this is Π_1. □

Lemma 3.6. *$I\Sigma_1$ proves: If $e \;{}_z e'$, then $Sat_0(z, e)$ iff $Sat_0(z, e')$.*

Proof. Suppose $e \;{}_z e'$ and $Sat_0(z, e)$. Then for some s, g, n, $PSat_0(s, g, n)$ and $s(z, e) = 1$. Let n' be greater than the largest value that e' assigns to any variable; let g' be the maximum of all the Gödel numbers of variables to which e' assigns a value; then certainly $PV(e', g', n')$. (Such n' and g' can be proven to exist in $I\Sigma_1$ since e.g. $max(e)$ can be defined by primitive recursion and so is defined by a formula Δ_1 in $I\Sigma_1$, making induction on *e* legitimate.) Let d be the restriction of e to variables in z; d is also the restriction of e' to variables in z: That is, $d \;{}_z e$ and $d \;{}_z e'$. By 3.4, for some s', we have $PSat_0(s', g', n')$ and $PV(d, g', n')$, so that $s'(z, d)$ is defined. But $s(z, e) = s(z, d)$, by 3.2, $= s'(z, d)$, by 3.3, and $= s'(z, e')$, by 3.2, again. So $s'(z, e') = 1$, and $Sat_0(z, e')$. □

Theorem 3.7. *$I\Sigma_1$ proves the Tarskian conditions for Sat_0 and Δ_0 formulas.*

Proof. The cases of identity and the connectives are easy. For example, suppose that $Sat_0(\ulcorner u = v \urcorner, e)$. Then for some s, g, n, $PSat_0(s, g, n)$ and $s(\ulcorner u = v \urcorner, e) = 1$. By definition, then, $val(u, e) = val(v, e)$. So suppose that $val(u, e) = val(v, e)$. Let d be the restriction of e to variables in u and v; let n be $val(u, e)$; let g be $\ulcorner u = v \urcorner$. By 3.4, there is an s such that $PSat_0(s, g, n)$; since $PV(d, g, n)$ (the value of a term must be greater than the value assigned to any variable that is contained in it!) and since $val(u, d) = val(v, d)$, by construction, we have $s(\ulcorner u = v \urcorner, d) = 1$. So $Sat_0(\ulcorner u = v \urcorner, d)$. But then, by 3.6, $Sat_0(\ulcorner u = v \urcorner, e)$, since $d \;{}_g e$.

So again, the hard case is that of bounded quantifiers, as one might expect. Let *z* be $\ulcorner (\forall w < v) u \urcorner$. Suppose $Sat_0(z, e)$, that is, that for some s, g, n, $PSat_0(s, g, n)$ and $s(z, e) = 1$. Then, by definition, $z \leq g$ and for any e' such that $PV(e', g, n)$ that agrees with e on $Fv(u)$, except possibly on w, and which is such that $e'(w)$ is defined and $< e'(v)$, we have $s(u, e') = 1$. Hence, we have $Sat_0(u, e')$, for all such e'. Now, we want to show that if d agrees with e on $Fv(u)$, except possibly on w, then if $d(w)$ is defined and $< d(v)$, $Sat_0(u, d)$. Let d' be the restriction of d to $Fv(u)$. Then $PV(d', g, n)$, and so d' is one of the e' just mentioned: Hence, $Sat_0(u, d')$. But then, by 3.6, $Sat_0(u, d)$.

Conversely, suppose that, if e' agrees with e on $Fv(u)$, except possibly for w, then $Sat_0(u, e')$. We want to show that $Sat_0(z, e)$. Let g and n be so chosen that $PV(e, g, n)$ and $z \leq g$. By 3.4, for some s, $PSat_0(s, g, n)$. Suppose now that d agrees with e on $Fv(u)$, except possibly for w, and that $d(w)$ is defined and $< d(v)$. We want to show that $s(u, d) = 1$. But d is one of the e' mentioned in the original supposition; so $Sat_0(u, d)$. Hence, for some s', g', n', $PSat_0(s', g', n')$ and $s'(u, d) = 1$. But then, by 3.3, $s(u, d) = s'(u, d) = 1$, so we are done. □